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THEORETICAL INVESTIGATION OF THE RADIATION CHARACTERISTICS OF AN ANTENNA.

By George W. Pierce.

VOLUME 52.

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1. Introduction.— For the proper design of a radiotelegraphic transmitting station it is important to know the radiation characteristics of different types of antenna.

For example, if a flat-top antenna is to be employed, the question arises as to what is the best relation of the length of the horizontal part to the length of the vertical part, when the excitation is to be produced by a given type of generator. It may be known in a general way that the greater the vertical length, the greater the radiation resistance; it may also be known that the greater the horizontal length of the flat-top the greater the capacity of the antenna will be, and the greater will be the amount of current that can be made to flow from certain types of generator. Now these two quantities, radiation resistance and applied current, are both factors in determining the out-put from the antenna.

For a given generator, with known characteristics, the problem of getting the greatest output of high frequency energy is a problem in the determination of the maximum value of the product of current square and radiation resistance of the antenna.

But this is not the whole problem, for there comes also into consideration the question as to how much of the radiated energy is radiated by the horizontal flat-top in what may be a useless direction.

Again, of the energy radiated from the vertical part of the antenna, how much of it contributes to the electric and magnetic forces on the horizon, where the receiving station is situated?

For the solution of these various problems it is important to know the radiation characteristics of the antenna in the form of certain functional relations. These relations should be known even when inductance is added at the base of the antenna for providing coupling or for increasing the wavelength to adapt it to the generator. These quantities should be known theoretically, since the ordinary measurements of these quantities do not permit us to distinguish radiation

that is useful from the useless radiation as heat losses and from the radiation in useless directions.

It is the purpose of this paper to give a treatment of this problem. Such a treatment is, so far as I know, up to the present entirely lacking, but the method here employed is that developed by Abraham [1] in a very remarkable paper entitled *Funkentelegraphie und Elektrodynamik*. In that paper, Abraham obtained theoretically the characteristics of a straight oscillator vibrating with its natural fundamental and harmonic frequencies. The present work is an extension of Abraham's method to the much more difficult problem of an antenna with a flat-top and with added inductance at the base.

2.   Inadequacy of the Conception of an Antenna as a Doublet.— Apart from the brilliant investigation by Abraham, all other attempts at the treatment of the radiation from an antenna assume that the antenna is a Hertzian Doublet.   This is only a very crude approximation to the facts, for *the derivation of the electromagnetic field about a doublet assumes that the length of the doublet is negligible in comparison with a quantity that is itself negligible in comparison with the wavelength.*

Hence, the doublet theory will apply in all of its essentials to an antenna, only provided the length of the antenna is not greater than one ten thousandth of the wavelength emitted.   Of course, it may be that at great distances from the oscillator, the theory that it is a doublet may not introduce any large errors into certain problems such as the propagation over the surface of the earth; but the present treatment shows that the doublet theory does introduce large errors into computations of such quantities as the electric and magnetic field intensities and the radiation resistance of an antenna.   It seems probable that other problems also should be revised in such a way as to replace the conception of the antenna as a doublet by the view of it as an oscillator that has a length comparable with one quarter of the wavelength.

3.   Method of the Present Investigation.— In the present investigation, a doublet of infinitesimal length is assumed *at each point of the antenna*.   This is the device used by Abraham.   These elementary doublets are free from the objection regarding their lengths, as they are of infinitesimal lengths, while the wavelength is that due to the

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1 M. Abraham: Physikalische Zeitschrift, **2**, 329–334 (1901).

whole antenna and therefore is enormously large in comparison with the lengths of the elemental doublets. The electric and magnetic forces due to each of the doublets is determined at a distance point and is summed up for all of the doublets of the antenna, *with strict regard to the difference of phase due to the different locations of the different doublets.* Such a process performed for all points of a distant sphere surrounding the antenna gives the total electric and magnetic forces at all points on the sphere. Then by integrating Poynting's Vector over the entire sphere, we obtain the total power radiated, and from this we compute the radiation resistance and other characteristics of the antenna.

The effect due to the vertical portion of the antenna and to the horizontal flat-top portion are computed separately, so as to give information as to how much energy is radiated with its electric force vertical to the horizon and how much parallel to the horizon.

In deciding as to the proper distribution of the elemental doublets along the antenna, the form of the current curve from point to point of the antenna is assumed independently. This process is not entirely above reproach, because Maxwell's equations, if they could be properly applied to the problem, would themselves give the distribution that is consistent with the applied electromotive force at the base of the antenna and with the shape and form of the antenna. This step of accurately deriving the distribution is, however, at the present time not possible of mathematical execution.

The distribution here assumed for the current in the antenna, as a function of the time and of the position along the antenna, is a generalization of the distribution assumed by Abraham, and is given in the next section.

4. Assumed Current Distribution.— The form of antenna to which the whole discussion is devoted is illustrated in Figure 1, and consists of a vertical portion of length a and a horizontal flat-top portion of length b. These quantities a and b may have any relative values whatever.

At the base of the antenna is an arbitrary inductance L for varying the wavelength.

The current at any point P$'$ of the antenna is assumed to be given by the equation

$$i = I \sin \frac{2\pi c}{\lambda} t \cdot \sin \frac{2\pi}{\lambda} \left(\frac{\lambda_0}{4} - l \right). \tag{1}$$

where
- c = velocity of light,
- λ_0 = natural wavelength of the antenna without inductance,
- λ = the wavelength with the inductance,
- i = the current at the point P′,
- l = length measured along the antenna from the inductance to the point P′.
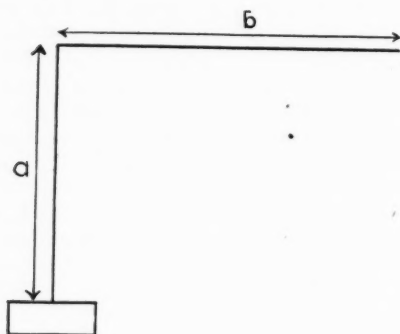

FIGURE 1.

The character of the assumed distribution is as follows: The factor $\sin \dfrac{2\pi c}{\lambda} t$ means that the current is sinusoidal in time at every point of the antenna, with the angular velocity

$$\omega = \frac{2\pi}{T} = \frac{2\pi c}{cT} = \frac{2\pi c}{\lambda}. \tag{2}$$

The meaning of the other factor

$$I \sin \frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - l\right) = J \ (\text{say}) \tag{3}$$

is illustrated in the diagrams (a), (b) and (c) of Figure 2.

If there is no inductance, $\lambda = \lambda_0$, and the factor becomes

$$J = I \cos \frac{2\pi l}{\lambda}. \tag{4}$$

This is illustrated in (a).

In the case with added inductance, $\lambda \neq \lambda_0$, and we must keep the general form of J given in equation (3). This equation for positive values of l gives the upper half of the diagram (b). When l is supposed
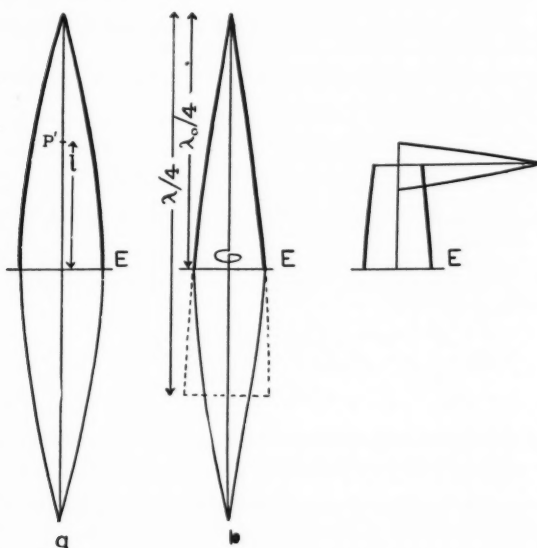

FIGURE 2.

negative the curves obtained continue along the dotted lines of (b) and do not give a figure symmetrical with the upper half. *To produce proper symmetry the absolute value of l must be employed in equation (1) when it is applied to the distribution of the image to take account of reflection.*

It is also to be carefully noted that when $l = 0$, equation (1) becomes

$$i_0 = I \sin \frac{\pi \lambda_0}{2\lambda} \sin \frac{2\pi c}{\lambda} t, \tag{5}$$

so the amplitude at the base of the antenna is

$$I_0 = I \sin \frac{\pi \lambda_0}{2\lambda} \tag{6}$$

Now, finally, when the antenna has a flat-top it is assumed that the top part of the antenna is bent over without any significant change in the magnitude of the current at the various points.

When the equation (1) is to be applied to the vertical portion of the antenna, we shall call

$$l = z', \tag{7}$$

where

z' = vertical distance from the ground of the point P′ on the antenna.

When the equation is to be applied to the horizontal part of the antenna, we shall call

$$l = a + x', \tag{8}$$

where

x' = distance along the horizontal part of the antenna to any point P″ on the flat-top.

The discussion will now be divided into several Parts: Part I. Electromagnetic Field Due to Vertical Portion of the Antenna; Part II. Field due to Horizontal Portion of the Antenna; Part III. The Mutual Term in Power Determination. Part IV. Computations of Radiation Resistance. Part V. Field Intensities and Summary.

PART I.

FIELD DUE TO VERTICAL PORTION OF ANTENNA.

5. Coördinates.— Let the origin of coördinates be at the point of connection of the antenna to the ground. Let the z-axis be vertical. About this vertical axis as polar diameter, let us construct a system
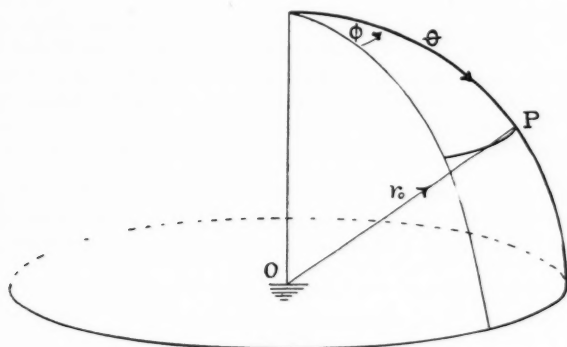

FIGURE 3.

of polar coördinates in which the position of any point P is given by its distance r_0 from the origin, and the angles θ and ϕ.

$\theta =$ the angle along meridional lines from the pole,
$\phi =$ the angle along parallels of latitude from a vertical plane of reference whose position is at present immaterial.

This system of coördinates with the positive directions of the angles indicated is given in Figure 3.

If z' is the vertical ordinate of any point P' on the vertical portion of the antenna, and r the distance from P' to P, and if the distance OP is large in comparison with z', we may write (see **Figure 4**)

$$r = r_0 - z'\cos \theta. \qquad (9)$$

6. Field Due to a Doublet at P'.— At a distant point P the electric and magnetic intensities due to a doublet of length dz' and charges e and $-e$ at P' is, by Hertz's theory,

$$dE_\theta = dH_\varphi = \frac{\sin \theta}{c^2 r_0} \ddot{f}\ (t - r/c), \qquad (10)$$

where

$f(t)$ = the moment of the doublet

$\qquad = e\ dz'$, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (11)

dE_θ = the electric intensity in electrostatic units, which is entirely in the direction of θ; that is, of the meridional lines;

$dH\phi$ = the magnetic intensity in electromagnetic units, which is entirely in the direction of the parallels of latitude.

r = distance P'P in centimeters,

c = velocity of light in centimeters per second.

The two dots over the f in (10) indicate the second time derivative.

In writing equation (10), the slight difference in the direction of the perpendicular to r from the direction of the perpendicular to r_0
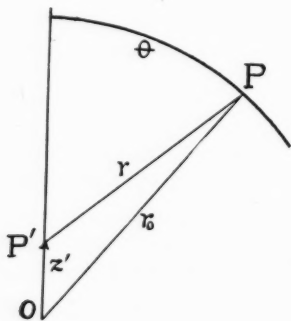

FIGURE 4.

is neglected in view of the largeness of r_0 in comparison with the length z' measured on the antenna.

Also the r which should occur in the denominator of (10) has been

replaced by r_0, which can be done without appreciable error for large values of r. The same substitution cannot be made in the argument of f in (10), for there r determines the phase of the oscillation, and this phase changes through an angle of π for a half wavelength, independent of the distance from the origin.

7. Expression of the Field in Terms of Current.

— We shall next express the moment of the doublet and the intensities of the field in terms of the current i at the point z'. To do this we shall think of the current as delivering a charge $+ e$ to one end of the element of length dz' and a change $- e$ to the other end of dz' in a certain time. A neighboring doublet has a different current and delivers different charges $+ e_1$ and $- e_1$ partly counteracting the charges of the given doublet, and leaving just the charge $e - e_1$ that actually occurs on the wire. This is represented in Figure 5.

FIGURE 5.

With this view of the case

$$i = \dot{e},$$

and

$$\ddot{f}(t) = \ddot{e}\, dz' = \frac{\partial i}{\partial t}\, dz'. \tag{12}$$

Whence, by substituting the value of i from equation (1) into equation (12) we shall have, in view of (7) and (9)

$$dE_\theta = dH_\phi = \frac{2\pi I \sin\theta}{\lambda c r_0} \cos \frac{2\pi}{\lambda}\, (ct - r_0 - z'\cos\theta)\cdot$$
$$\sin\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - z'\right) dz'. \tag{13}$$

By integrating this expression from $z' = 0$ to $z' = a$, we obtain the electric and magnetic intensities at the point P due to direct transmission from the vertical portion of the antenna. Indicating this integration, we have
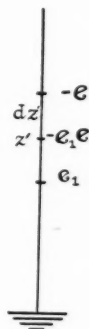
$$E_\theta = H_\phi = \frac{2\pi\, I \sin \theta}{\lambda c r_0} \int_o^a \cos \frac{2\pi}{\lambda}\, (ct - r_0 - z' \cos \theta) \cdot$$

$$\sin \frac{2\pi}{\lambda} \left(\frac{\lambda_0}{4} - z' \right) dz'. \quad (14)$$

By reflection from the earth, which we shall regard as a perfect reflector, we have intensities that must be added to the above. These intensities may be obtained by considering the radiation to come from an image point at a distance z' below the surface. The effect of this is obtained by changing the sign of the z' in the cosine term of equation (14), but as was pointed out in section **4** the sign of z' in the sine term must remain. We obtain thus for the intensities due to the reflected wave emitted by the vertical portion of the antenna the value

$$E_\theta = H_\phi = \frac{2\pi\, I \sin \theta}{\lambda c r_0} \int_o^a \cos \frac{2\pi}{\lambda}\, (ct - r_0 + z' \cos \theta) \cdot$$

$$\sin \frac{2\pi}{\lambda} \left(\frac{\lambda_0}{4} - z' \right) dz'. \quad (15)$$

Adding the equation (15) for the reflected intensities to the direct intensities of (14), remembering that if A and B are any two angles

$$\cos (A - B) + \cos (A + B) = 2 \cos A \cos A, \quad (16)$$

we obtain for the total intensities at P the equation

$$E_\theta = H_\phi = \frac{4\pi\, I \sin \theta}{\lambda c r_0} \cos \frac{2\pi}{\lambda}\, (ct - r_0) \int_o^a \cos \left(\frac{2\pi z'}{\lambda} \cos \theta \right) \cdot$$

$$\sin \frac{2\pi}{\lambda} \left(\frac{\lambda_0}{4} - z' \right) dz', \quad (17)$$

which resolves into

$$E_\theta = H_\phi = \frac{4\pi\, I \sin \theta}{\lambda c r_0} \cos \frac{2\pi}{\lambda}\, (ct - r_0) \Bigg[$$

$$\sin \frac{\pi \lambda_0}{2\lambda} \int_o^a \cos \frac{2\pi z' \cos \theta}{\lambda} \cos \frac{2\pi\, z'}{\lambda}\, dz'$$

$$- \cos \frac{\pi\, \lambda_0}{2\, \lambda} \int_o^a \cos \frac{2\pi z' \cos \theta}{\lambda} \sin \frac{2\pi\, z'}{\lambda}\, dz' \Bigg]. \quad (18)$$

This expression may be integrated by the formulas 360 and 361 of B. O. Pierce's *Short Table of Integrals* and gives

$$E_\theta = H_\phi = \frac{2 I}{c r_0 \sin \theta} \cos \frac{2\pi}{\lambda} (ct - r_0) \Big\}$$

$$\cos B \cos (A \cos \theta) - \sin B \cos \theta \sin (A \cos \theta) - \cos G \Big\} \quad (19)$$

where

$$\left. \begin{aligned} B &= \frac{2\pi b}{\lambda} \\ A &= \frac{2\pi a}{\lambda} \\ G &= \frac{\pi \lambda_0}{2\lambda} = A + B \end{aligned} \right\} \quad (20)$$

The quantity b, which is the length of the flat top, gets into (20) and (19) by reason of the fact that $a + b =$ the whole length of the antenna, so that

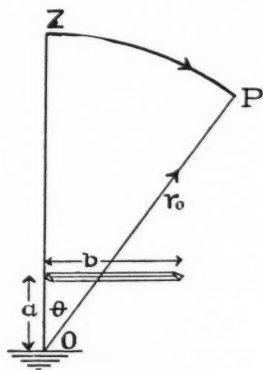$$\lambda_0 = 4 (a + b). \quad (21)$$

FIGURE 6.

Equation (19), with the notation of equations (20) and (21) is the general equation for the electric and magnetic Intensities at any point P,

due to the whole vertical part of the antenna. In this formula, referring to Figure 6,

r_0 = the distance OP in cm.,
θ = the zenith angle ZOP,
b = length of the horizontal flat top in meters,
a = length of vertical part of antenna, in meters,
$\lambda_0 = 4 (a + b)$ = natural wave length in meters,
λ = wave length in meters actually emitted, and differing from λ_0
    by virtue of the added inductance,
I_0 = amplitude of current in absolute electrostatic units at the base
    of the antenna and related to I by the equation,

$$I_0 = I \sin \frac{\pi \lambda_0}{2\lambda} \cdot$$

We shall reserve comment on this equation until after investigation of other characteristics of the radiation. See Part IV.

8. Total Power Radiated from the Vertical Part of the Antenna. — Having obtained in equation (19) the electric and magnetic intensities at any required point at a distance from the antenna, we shall next compute the total power radiated from the vertical part of the antenna, and shall then obtain its radiation resistance.

Since E_θ and H_ϕ are perpendicular to one another and perpendicular to r_0, we have, according to Poynting's theorem for the power radiated in the direction of r_0 through an element of surface dS perpendicular to r_0 the quantity

$$dp = \frac{c}{4\pi} E_\theta H_\phi \, dS. \tag{22}$$

Let the element of surface be an elemental zone on the surface of the sphere, then

$$dS = 2\pi r_0^2 \sin \theta d\theta \tag{23}$$

This quantity, together with the values of E_θ and H_ϕ from (19), substituted in (22) and properly integrated, gives for the total power radiated through the whole hemisphere above the earth's surface, the value in ergs per second following:

$$p = \frac{2I^2}{c} \cos^2 \left\{ \frac{2\pi}{\lambda} (ct - r_0) \right\} \left[\cos^2 B \int_o^{\pi/2} \frac{\cos^2 (A \cos \theta) \, d\theta}{\sin \theta} \right.$$

$$+ \sin^2 B \int_o^{\pi/2} \frac{\cos^2 \theta \sin^2 (A \cos \theta) \, d\theta}{\sin \theta} + \cos^2 G \int_o^{\pi/2} \frac{d\theta}{\sin \theta}$$

$$- 2 \sin B \cos B \int_o^{\pi/2} \frac{\cos \theta \sin (A \cos \theta) \cos (A \cos \theta) \, d\theta}{\sin \theta}$$

$$- 2 \cos B \cos G \int_o^{\pi/2} \frac{\cos (A \cos \theta) \, d\theta}{\sin \theta}$$

$$+ 2 \sin B \cos G \int_o^{\pi/2} \frac{\cos \theta \sin (A \cos \theta) \, d\theta}{\sin \theta} \right]. \tag{24}$$

This equation when integrated gives the power radiated from the vertical part of the antenna. The integration is a tedious operation, and is given in the next section, which may be omitted by readers not interested in the mathematical processes involved. The result of the integration is found in Section 10.

9. The Integration of Equation (24). — By the use of such trigonometric equations as

$$\cos^2 x = \frac{1 + \cos 2x}{2},$$

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

the squares of sines and cosines in the integrands of (24) may be avoided, and equation (24) written

$$p = \frac{2 I^2}{c} \cos^2 \left\{ \frac{2\pi}{\lambda} (ct - r_0) \right\} \left[\left(\frac{1}{2} + \cos^2 G \right) \int_o^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta} \right.$$

$$+ \frac{\cos 2 B}{2} \int_o^{\pi/2} \frac{\cos (2 A \cos \theta) \, d\theta}{\sin \theta} - \frac{\sin^2 B}{2} \int_o^{\pi/2} \sin \theta \, d\theta$$

$$+ \frac{\sin^2 B}{2} \int_o^{\pi/2} \sin \theta \cos (2 A \cos \theta) \, d\theta$$

$$- \frac{\sin 2 B}{2} \int_o^{\pi/2} \frac{\cos \theta \sin (2 A \cos \theta) \, d\theta}{\sin \theta}$$

$$- 2 \cos B \cos G \int_o^{\pi/2} \frac{\cos (A \cos \theta) \, d\theta}{\sin \theta}$$

$$+ 2 \sin B \cos G \int_o^{\pi/2} \frac{\cos \theta \sin (A \cos \theta) \, d\theta}{\sin \theta} \right]. \tag{25}$$

The third and fourth terms may be integrated directly. In the other terms let us introduce a change of variable as follows:

Let

$$u = \cos \theta$$

$$d\theta = \frac{-\, du}{\sin \theta},$$

then

$$\int_o^{\pi/2} \frac{d\theta}{\sin \theta} = \int_1^o \frac{-\, du}{1 - u^2} = \frac{1}{2} \int_o^1 \left(\frac{1}{1 + u} + \frac{1}{1 - u} \right) du$$

$$= \frac{1}{2} \int_o^1 \frac{du}{1 + u} + \frac{1}{2} \int_{-1}^o \frac{du}{1 + u} = \frac{1}{2} \int_{-1}^{+1} \frac{du}{1 + u}. \quad (26)$$

With this operation as a model, two of the other integrals of (25) may be written, respectively

$$\int_o^{\pi/2} \frac{\cos (2\, A \cos \theta)\, d\theta}{\sin \theta} = \frac{1}{2} \int_{-1}^{+1} \frac{\cos (2\, Au)\, du}{1 + u}, \quad (27)$$

$$\int_o^{\pi/2} \frac{\cos (A \cos \theta)\, d\theta}{\sin \theta} = \frac{1}{2} \int_{-1}^{+1} \frac{\cos (Au)\, du}{1 + u}. \quad (28)$$

Another of the integrals, examined in more detail, gives

$$\int_o^{\pi/2} \frac{\cos \theta \sin (2\, A \cos \theta)\, d\theta}{\sin \theta}$$

$$= \int_o^1 \frac{u \sin (2\, Au)\, du}{1 - u^2}$$

$$= \frac{1}{2} \int_o^{+1} \left(\frac{1}{1 - u} - \frac{1}{1 + u} \right) \sin (2\, Au)\, du$$

$$= -\frac{1}{2} \int_o^1 \frac{\sin (2\, Au)\, du}{1 + u} + \frac{1}{2} \int_o^{-1} \frac{\sin (2\, Au)\, du}{1 + u}$$

$$= -\frac{1}{2} \int_{-1}^{+1} \frac{\sin (2\, Au)\, du}{1 + u}. \quad (29)$$

Similarly, the remaining integral becomes

$$\int_o^{\pi/2} \frac{\cos \theta \sin (A \cos \theta)\, d\theta}{\sin \theta} = -\frac{1}{2} \int_{-1}^{+1} \frac{\sin (Au)\, du}{1 + u}. \quad (30)$$

Returning now to equation (25), we shall integrate the third and fourth terms, setting them first, and shall substitute (26) to (30) for the other terms, obtaining

$$
\begin{aligned}
p = \frac{2\,I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}\,(ct - r_0)\right\} & \left[-\frac{\sin^2 B}{2} + \frac{\sin^2 B}{2}\frac{\sin 2\,A}{2\,A} \right. \\
& + \left(\frac{1}{2} + \cos^2 G\right)\frac{1}{2}\int_{-1}^{+1}\frac{du}{1 + u} \\
& + \frac{\cos 2\,B}{4}\int_{-1}^{+1}\frac{\cos(2\,Au)\,du}{1 + u} + \frac{\sin 2\,B}{4}\int_{-1}^{+1}\frac{\sin(2\,Au)\,du}{1 + u} \\
- \cos G\left\{\cos B\int_{-1}^{+1}\frac{\cos(Au)\,du}{1 + u}\right. & \left.\left. + \sin B\int_{-1}^{+1}\frac{\sin(Au)\,du}{1 + u}\right\}\right]. \quad (31)
\end{aligned}
$$

Let us now write

$$
\begin{aligned}
\gamma &= 2\,A\,(1 + u), \\
2\,Au &= \gamma - 2A, \\
du &= \frac{d\gamma}{2\,A}, \\
\frac{du}{1 + u} &= \frac{d\gamma}{\gamma};
\end{aligned}
$$

then the second and third integrals of (31) become

$$
\begin{aligned}
\frac{\cos 2\,B}{4}&\int_{-1}^{+1}\frac{\cos(2\,Au)\,du}{1 + u} + \frac{\sin 2\,B}{4}\int_{-1}^{+1}\frac{\sin(2\,Au)\,du}{1 + u} \\
&= \frac{\cos 2\,B}{4}\int_{o}^{4A}\{\cos\gamma\cos 2\,A + \sin\gamma\sin 2\,A\}\frac{d\gamma}{\gamma} \\
&\qquad + \frac{\sin 2\,B}{4}\int_{o}^{4A}\{\sin\gamma\cos 2\,A - \cos\gamma\sin 2\,A\}\frac{d\gamma}{\gamma} \\
&= \frac{\cos(2\,A + 2\,B)}{4}\int_{o}^{4A}\frac{\cos\gamma\,d\gamma}{\gamma} + \frac{\sin(2\,A + 2\,B)}{4}\int_{o}^{4A}\frac{\sin\gamma\,d\gamma}{\gamma} \\
&= \frac{\cos 2\,G}{4}\int_{o}^{4A}\frac{\cos\gamma}{\gamma}\,d\gamma + \frac{\sin 2\,G}{4}\int_{o}^{4A}\frac{\sin\gamma}{\gamma}\,d\gamma.
\end{aligned}
$$

In like manner, the last line of (31) becomes

$$
-\cos^2 G\int^{2A}\frac{\cos\gamma}{\gamma}\,d\gamma - \cos G\sin G\int_{o}^{2A}\frac{\sin\gamma}{\gamma}\,d\gamma. \quad (33)
$$

Let us now decompose the coefficient of the first integral of (31) as follows:

$$\frac{1}{4} + \frac{\cos^2 G}{2} = \frac{1}{4} + \cos^2 G - \frac{\cos^2 G}{2}$$

$$= \frac{1}{4} - \frac{1 + \cos 2 G}{4} + \cos^2 G$$

$$= -\frac{\cos 2 G}{4} + \cos^2 G.$$

Then the whole equation (31) may be written

$$p = \frac{2\,I^2}{c} \cos^2 \left\{ \frac{2\pi}{\lambda}\,(ct - r_0) \right\} \left[-\frac{\sin^2 B}{2} + \frac{\sin^2 B \sin 2 A}{4\,A} \right.$$
$$- \frac{\cos 2 G}{4} \int_0^{4A} \frac{(1 - \cos \gamma)\,d\gamma}{\gamma} + \frac{\sin 2 G}{4} \int_0^{4A} \frac{\sin \gamma}{\gamma}\,d\gamma$$
$$\left. + \cos^2 G \int_0^{2A} \frac{(1 - \cos \gamma)\,d\gamma}{\gamma} - \frac{\sin 2 G}{2} \int_0^{2A} \frac{\sin \gamma}{\gamma}\,d\gamma \right]. \quad (34)$$

The various integrals may now be obtained by expanding in series and integrating term by term.   This gives

$$p = \frac{2\,I^2}{c} \cos^2 \left\{ \frac{2\pi}{\lambda}\,(ct - r_0) \right\} \left[\frac{\sin^2 B}{2}\left(\frac{\sin 2A}{2A} - 1 \right) \right.$$

$$- \frac{\cos 2G}{4} \left\{ \frac{(4A)^2}{2!\,2} - \frac{(4A)^4}{4!\,4} + \frac{(4A)^6}{6!\,6} - \cdots \right\}$$

$$+ \frac{1 + \cos 2G}{2} \left\{ \frac{(2A)^2}{2!\,2} - \frac{(2A)^4}{4!\,4} + \frac{(2A)^6}{6!\,6} - \cdots \right\}$$

$$+ \frac{\sin 2G}{4} \left\{ \ 4A \ - \frac{(4A)^3}{3!\,3} + \frac{(4A)^5}{5!\,5} - \cdots \right\}$$

$$\left. - \frac{\sin 2G}{2} \left\{ \ 2A \ - \frac{(2A)^3}{3!\,3} + \frac{(2A)^5}{5!\,5} - \cdots \right\} \right]. \quad (35)$$

Let us now eliminate B from the first terms of this equation, by substituting $B = G - A$, obtaining

$$\frac{\sin{}^2 B}{2}\left(\frac{\sin 2A}{2A} - 1\right) = \frac{1 - \cos 2B}{4}\left(\frac{\sin 2A}{2A} - 1\right)$$

$$= \left\{\frac{1}{4} - \frac{\cos\left(2G - 2A\right)}{4}\right\}\left(\frac{\sin 2A}{2A} - 1\right)$$

$$= -\frac{1}{4} + \frac{\cos 2G \cos 2A}{4} + \frac{\sin 2G \sin 2A}{4}$$

$$+ \frac{\sin 2A}{8A} - \frac{\cos 2G \sin 4A}{16A}$$

$$- \frac{\sin 2G}{4}\frac{1 - \cos 4A}{4A}. \tag{36}$$

If now we expand in series the quantities involving A in (36) and substitute in (35), we obtain, if

$$\left.\begin{array}{l} k = 2A \\ q = 2G \end{array}\right\} \tag{37}$$

$$p = \frac{2I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\left[\frac{1}{4}\left\{-\frac{k^2}{3!} + \frac{k^4}{5!} - \frac{k^6}{7!} + \cdots\right\}\right.$$

$$+ \frac{1}{4}\left\{\frac{2k^2}{2!2} - \frac{2k^4}{4!4} + \frac{2k^6}{6!6} - \cdots\right\}$$

$$+ \frac{\cos q}{4}\left\{1 \quad - \frac{k^2}{2!} + \frac{k^4}{4!} - \frac{k^6}{6!} + \cdots\right\}$$

$$+ \frac{\sin q}{4}\left\{k \quad - \frac{k^3}{3!} + \frac{k^5}{5!} - \frac{k^7}{7!} + \cdots\right\}$$

$$- \frac{\cos q}{4}\left\{1 \quad - \frac{(2k)^2}{3!} + \frac{(2k)^4}{5!} + \cdots\right\}$$

$$- \frac{\sin q}{4}\left\{\frac{2k}{2!} \quad - \frac{(2k)^3}{4!} + \frac{(2k)^5}{6!} - \cdots\right\}$$

$$- \frac{\cos q}{4}\left\{\frac{(2k)^2}{2!2} \quad - \frac{(2k)^4}{4!4} + \frac{(2k)^6}{6!6} - \cdots\right\}$$

$$+ \frac{\cos q}{2}\left\{\frac{k^2}{2!2} \quad - \frac{k^4}{4!4} + \frac{k^6}{6!6} - \cdots\right\}$$

$$+ \frac{\sin q}{4}\left\{2k \quad - \frac{(2k)^3}{3!3} + \frac{(2k)^5}{5!5} - \cdots\right\}$$

$$\left.- \frac{\sin q}{2}\left\{k \quad - \frac{k^3}{3!3} + \frac{k^5}{5!5} - \cdots\right\}\right]. \tag{38}$$

If now we add together the terms multiplied by sin q and those multiplied by cos q, and those not involving q, we have (on factoring out the $\frac{1}{4}$)

$$
\begin{aligned}
p = \frac{I^2}{2c} \cos^2 &\left\{ \frac{2\pi}{\lambda} (ct - r_0) \right\} \left\{ \left[\left\{ \frac{2+2}{3!2} k^2 - \frac{4+2}{5!4} k^4 + \frac{6+2}{7!6} k^6 - \right. \right. \right. \\
&\left. \cdots \right\} \\
+ \cos q &\left\{ -\frac{2^2 + 2^2 - 4}{3!2} k^2 + \frac{4^2 + 2^4 - 6}{5!4} k^4 - \right. \\
&\qquad \left. \frac{6^2 + 2^6 - 8}{7!6} k^6 + \frac{8^2 + 2^8 - 10}{9!8} k^8 - \cdots \right\} \\
+ \sin q &\left\{ -\frac{3^2 + 2^3 - 5}{4!3} k^3 + \frac{5^2 + 2^5 - 7}{6!5} k^5 - \right. \\
&\qquad \left. \frac{7^2 + 2^7 - 9}{8!7} k^7 + \frac{9^2 + 2^9 - 11}{10!9} k^9 - \cdots \right\} \cdot
\end{aligned}
$$

$$\tag{39}$$

Equation (39) gives the total power radiated by the vertical portion of the antenna into the hemisphere above the earth's surface. In this equation, the current factor I is in absolute c. g. s. electrostatic units, and the power p is in ergs per second.

It is convenient to change the current factor into amperes and the radiated power into watts, which can be done by multiplying the right hand side of (39) by 30 c. This is done, and the equation is rewritten in the next section.

10. Result of the Integration for Power Radiated from the Vertical Part of the Antenna. — By equation (39), when reduced to practical units, the total power radiated into the aërial hemisphere from the vertical part of the antenna may be written

$$
p = I^2 \cos^2 \left\{ \frac{2\pi}{\lambda} (ct - r_0) \right\} \left[R_1 - R_2 \cos q - R_3 \sin q \right]. \tag{40}
$$

where

$$R_1 = 15 \left\{ \frac{2+2}{3!2} k^2 - \frac{4+2}{5!4} k^4 + \frac{6+2}{7!6} k^6 - \cdots \right\}$$

$$R_2 = 15 \left\{ \frac{2^2 + 2^2 - 4}{3!2} k^2 - \frac{4^2 + 2^4 - 6}{5!4} k^4 + \frac{6^2 + 2^6 - 8}{7!6} k^6 - \cdots \right\}$$

$$R_3 = 15 \left\{ \frac{3^2 + 2^3 - 5}{4!3} k^3 - \frac{5^2 + 2^5 - 7}{6!5} k^5 + \frac{7^2 + 2^7 - 9}{8!7} k^7 - \cdots \right\}$$

(41)

$$q = \frac{\pi\lambda_0}{\lambda}$$

$$k = \frac{4\pi a}{\lambda}$$

(42)

$a = $ length of vertical part of antenna in same unit as λ

(e. g. meters),

$p^\cdot = $ radiated power in watts instantaneous value,

$$I = \frac{I_0}{\sin q/2},$$

(43)

where

$I_0 = $ amplitude of current at the base of antenna in amperes.

11. Radiation Resistance of Vertical Part of the Antenna.
— In equation (40) is given the power radiated from the vertical part of the antenna, on the assumption that radiation from the horizontal part of the antenna does not interfere with it.   It will be shown later in §14 et seq. how this interference is computed and allowed for. Accepting for the present the assumption of non-interference, we may obtain the radiation resistance of the vertical part of the antenna.

The radiation resistance is defined as the *time average of radiated power divided by the time average of the square of the current at the base of the antenna.*

In taking the time average of the power (40), it is to be noted that

the time average of $\cos^2 \left\{ \dfrac{2\pi}{\lambda} (ct - r_0) \right\}$ is $1/2$. The time average of

current square at the base of the antenna, by (1) is $\dfrac{1}{2} I^2 \sin^2 \dfrac{\pi\lambda_0}{2\lambda} =$

$\dfrac{1}{2} I^2 \sin^2 \left(\dfrac{q}{2}\right)$ Whence the radiation resistance becomes in ohms

$$R_\Omega = \dfrac{1}{\sin^2 \left(\dfrac{q}{2}\right)} \left\{ R_1 - R_2 \cos q - R_3 \sin q \right\}, \qquad (44)$$

in which R_1, R_2, R_3 and q have the values given in (41) and (42).

We shall later give tables of R_1, R_2, and R_3, that will reduce the calculations of R to very simple operations, and shall compare the results with calculations on the doublet hypothesis and with observations.

We shall, however, first investigate theoretically the radiation from the horizontal part of the antenna.   This is a problem of considerable mathematical difficulty but is capable of solution.

PART II.

FIELD DUE TO HORIZONTAL PORTION OF ANTENNA.

12. Introductory Notions. — To determine the electromagnetic field and radiation characteristics of the horizontal flat-top portion of the antenna, let the rectangular coördinates of any distant point P (Fig. 7) be x, y, z.
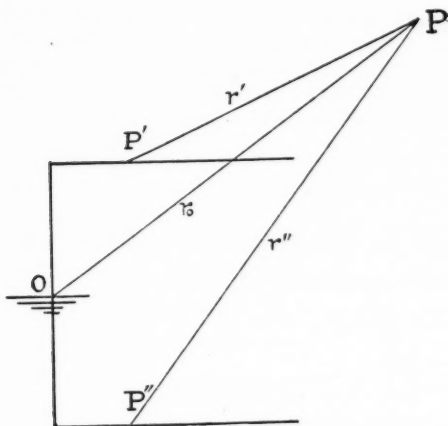

FIGURE 7.

And let the coördinates of any point P' on the flat-top be $x', 0, a$; the coördinates of the image point P'' be $x', 0, -a$.

Then the distance from the origin of coördinates to the distant point is

$$OP = r_0 = \sqrt{x^2 + y^2 + z^2}$$

The distances of the distant point from the point on the flat-top and its image respectively are

$$P' P = r' = \sqrt{(x - x')^2 + y^2 + (z - a)^2}.$$

and $$P'' P = r'' = \sqrt{(x - x')^2 + y^2 + (z + a)^2}.$$

Then $r' - r_0 = \sqrt{(x - x')^2 + y^2 + (z - a)^2} - \sqrt{x^2 + y^2 + z^2}.$

As an approximation, let us multiply by the sum of these radicals and divide by the approximate value of this sum for large values of r_0; namely, by $2r_0$, obtaining

$$r' = r_0 + \frac{x'^2 - 2xx' - 2za + a^2}{2r_0} \, , \tag{45}$$

$$r'' = r_0 + \frac{x'^2 - 2xx' + 2za + a^2}{2r_0} \, . \tag{46}$$

13. Determination of Electric and Magnetic Intensities due to Flat-top. — The values of r' and r'' in (45) and (46) may be replaced by r_0 in intensity factors, but not in phase terms, and give for the sum of the effects of a doublet at P' and another at P'' (the image doublet) the electric and magnetic intensities

$$dE_\psi = dH_\Sigma = \frac{\sin\psi}{r_0 c^2} \left\{ \ddot{f}_1 \left(t - r'/c \right) + \ddot{f}_2 \left(t - r''/c \right) \right\}, \tag{47}$$

where $f_1(t)$ and $f_2(t)$ are the moments of the two doublets respectively.
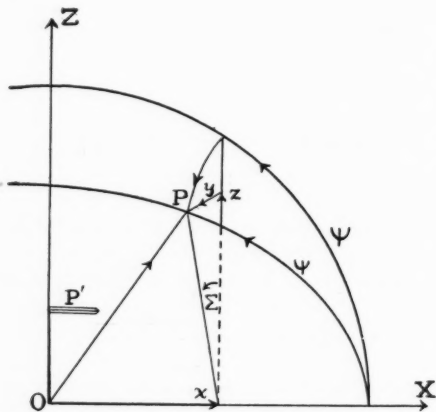

FIGURE 8.

The angles φ and Σ correspond to the angles θ and ϕ of Figure 3, except that the figure is turned on its side, so as to put the polar diameter

along the x-axis instead of the z-axis. This arrangement is shown in Figure 8. The plane of the zero value of Σ is now to be fixed as the plane of the x and z-axes.

Now using the current distribution of equation (1), we must replace l by $a + x'$, which gives, when treated as (12) was treated,

$$\ddot{f}_1 = \frac{2\pi c I}{\lambda} \cos\left\{ \frac{2\pi}{\lambda}\left((ct - r_0 - \frac{x'^2 - 2x'x - 2za + a^2}{2r_0}) \right) \right\}$$
$$\sin\left\{ \frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a - x'\right) \right\} dx'. \tag{48}$$

The fictitive current at P'' is just equal and opposite to that at P', with, however, a different distance from the point P, so we may write

$$\ddot{f}_2 = -\frac{2c\pi I}{\lambda} \cos\left\{ \frac{2\pi}{\lambda}\left(ct - r_0 - \frac{x'^2 - 2xx' + 2za + a^2}{2r_0}\right) \right\}$$
$$\sin\left\{ \frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a - x'\right) \right\} dx'. \tag{49}$$

Whence by addition, employing the trigonometric relation

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2 \sin\alpha \sin\beta,$$

equation (47) becomes

$$dE_\psi = dH_\Sigma = -\frac{4\pi I \sin\psi}{r_0 c \lambda} \sin\left\{ \frac{2\pi}{\lambda}\left(ct - r_0 - \frac{x'^2 - 2xx' + a^2}{2r_0}\right) \right\}$$
$$\left[\sin\frac{2\pi a z}{\lambda r_0} \sin\left\{ \frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a - x'\right) \right\} dx' \right].$$

In this equation we may as usual replace $\frac{2\pi a}{\lambda}$ by A. Also we may make an approximation as follows: For large values of r_0

$$\frac{x'^2 - 2xx' + a^2}{2r_0} = -\frac{xx'}{r_0} = -x' \cos\psi.$$

In making this approximation the neglected term is $\frac{x'^2 + a^2}{2r_0}$, and

this is to be neglected even in the phase angle, because its value is absolutely small.   We have then

$$dE_\psi = dH_\Sigma = -\frac{4\pi I \sin \psi}{r_0 c\lambda}\ \sin \frac{Az}{r_0}\ \sin \left\{ \frac{2\,\pi}{\lambda}\ (\mathrm{c}t - r_0 + x' \cos \psi)\ \right\}$$

$$\left[\sin \left\{ \frac{2\,\pi}{\lambda}\!\left(\frac{\lambda_0}{4} - a - x'\right)\right\} dx'\right]. \qquad (50)$$

This equation may be shortened up by writing

$$\tau = \frac{2\pi}{\lambda}\ (\mathrm{c}t - r_0) \qquad\qquad (51)$$

and

$$\mathrm{B} = \frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a\right) = \frac{2\pi b}{\lambda}. \qquad\qquad (52)$$

To obtain the total electric and magnetic intensities due to the flat-top, the equation (50) must be integrated for all the doublets and their images between the limits

$$x' = 0 \text{ and } x' = b$$

where b is the length of the flat-top.   This integration is expressed in the following equation.

$$E_\psi = H_\Sigma = -\frac{4\pi I \sin \psi}{r_0 c\lambda}\ \sin \frac{Az}{r_0}\int^{b} \sin \left(\tau + \frac{2\pi x'}{\lambda}\ \cos \psi\right) \sin$$

$$\left(\mathrm{B} - \frac{2\pi x'}{\lambda}\right) dx'. \qquad (53)$$

To perform the integration let us introduce a change of variable by putting

$$s = \mathrm{B} - \frac{2\pi x'}{\lambda} \qquad\qquad\qquad \text{then } dx = -\frac{\lambda}{2\pi}\,ds$$

and the limits of integration become

$$\text{for } x' = 0, \quad s = \mathrm{B}, \qquad\qquad \text{for } x' = b, \quad s = 0.$$

Equation (53) then becomes

$$E_\psi = H_\Sigma = \frac{2I \sin\psi}{r_0 c} \sin\frac{Az}{r_0} \int_B^o \sin(\tau + B\cos\psi - s\cos\psi)\sin s\, ds$$

$$= \frac{2\,I \sin\psi}{r_0 c} \sin\frac{Az}{r_0}\left[\sin(\tau + B\cos\psi)\int_B^o \cos(s\cos\psi)\sin s\, ds \right.$$

$$\left. - \cos(\tau + B\cos\psi)\int_B^o \sin(s\cos\psi)\sin s\, ds \right]\cdot \quad (54)$$

The expressions of this equation may be integrated by the use of formulas 360 and 359 of B. O. Peirce's Tables and give

$$E_\psi = H_\Sigma = \frac{2\,I}{r_0 c \sin\psi} \sin\frac{Az}{r_0}\Big\{ \sin(\tau + B\cos\psi)\Big[-\cos s\cos(s\cos\psi)$$

$$-\cos\psi\sin s\sin(s\cos\psi)\Big]_B^0$$

$$-\cos(\tau + B\cos\psi)\Big[\cos\psi\sin s\cos(s\cos\psi)$$

$$-\cos s\sin(s\cos\psi)\Big]_B^0$$

$$= \frac{2\,I}{r_0 c \sin\psi}\sin\frac{Az}{r_0}\Big[\sin(\tau + B\cos\psi)\Big\{ -1 + \cos B\cos$$

$$\Big(B(\cos\psi)\Big)$$

$$+ \cos\psi\sin B\sin(B\cos\psi)\Big\}$$

$$+ \cos(\tau + B\cos\psi)\cdot\Big\{ \cos\psi\sin B\cos(B\cos\psi)$$

$$-\cos B\sin(B\cos\psi)\Big\}\Big]$$

$$= \frac{2\,I}{r_0 c \sin\psi}\sin\frac{Az}{r_0}\Big[\sin\tau\Big\{ \cos B - \cos(B\cos\psi)\Big\}$$

$$+ \cos\tau\Big\{ \cos\psi\sin B - \sin(B\cos\psi)\Big\}\Big]\cdot \quad (55)$$

Equation (55) gives the electric and magnetic intensities due to the flat-top at any distant point whose coördinates are

r_0 = distance of the point from the origin,

z = vertical height of the point above the earth's surface,

ψ = angle between r_0 and the x-axis; this x-axis being parallel to the flat-top.

The quantities A, B, and τ are defines by equations (20) and (51). We shall next discuss the total power radiated from the antenna.

14.   Concerning Power Radiated from the Total Antenna.— It is to be noticed that the electric and magnetic intensities due to the flat-top of the antenna and those intensities due to the vertical portions of the antenna are directed along the meridional and latitudinal lines of two systems of polar coördinates with their poles one
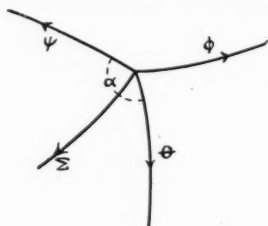

FIGURE 9.

quadrant apart. This does not make the respective intensities perpendicular to each other, and it becomes necessary to resolve one set of these intensities along and perpendicular to the other set of intensities.   At a given point on the sphere about the origin of coordinates, the quantities ϕ, θ, Σ, and ψ are oriented in a manner represented in Figure 9.

If we let

$$a = \text{angle between } \psi \text{ and } \theta$$

then also

$$a = \text{angle between } \phi \text{ and } \Sigma.$$

It is also apparent that

Angle between Σ and $\theta = a - \dfrac{\pi}{2}$

Angle between ψ and $\phi = \dfrac{3\pi}{2} - a$

Let us now resolve E_ψ and H_Σ into components along θ and perpendicular thereto (that is, along ϕ) obtaining for the θ-components

$$E_{\psi,\theta} = E_\psi \cos a$$

$$H_{\Sigma,\theta} = H_\Sigma \cos\left(a - \frac{\pi}{2}\right) = H_\Sigma \sin a.$$

and for the ϕ-components

$$E_{\psi,\varphi} = E_\psi \cos\left(\frac{3\pi}{2} - a\right) = -E_\psi \sin a$$

$$H_{\Sigma,\varphi} = H_\Sigma \cos a.$$

Adding these quantities to the corresponding components of the intensities due to the vertical part of the antenna, we obtain for the total intensities, which are designated by primes, the values

$$\begin{aligned}
E'_\theta &= E_\theta + E_\psi \cos a, \\
E'_\varphi &= -E_\psi \sin a, \\
H'_\theta &= H_\Sigma \sin a \\
H'_\varphi &= H_\varphi + H_\Sigma \cos a.
\end{aligned}$$

All of these intensities are perpendicular to r_0. To get the power radiated through an element of surface dS perpendicular to r_0, we may make use of Poynting's vector, in the form

$$dp = \frac{c}{4\pi}\left(\mathbf{E'} \times \mathbf{H'}\right) dS$$

where the *cross* between the vectors means the vector-product. This vector-product, expanded, gives

$$\begin{aligned}
dp &= \frac{c}{4\pi}\left(E'_\theta H'_\varphi - E'_\varphi H'_\theta\right) dS \\
&= \frac{c}{4\pi}\Big(E_\theta H_\varphi + E_\psi H_\Sigma \cos^2 a + H_\varphi E_\psi \cos a + E_\theta H_\Sigma \cos a + \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad E_\psi H_\Sigma \sin a\Big) dS \\
&= \frac{c}{4\pi}\left(E_\theta H_\varphi + E_\psi H_\Sigma + 2\cos a\, E_\theta H_\psi\right) dS. \qquad (56)
\end{aligned}$$

We have already found the first term of this power and have obtained its integral all over the aërial hemisphere. This integral we

have called *the power radiated from the vertical part* of the antenna. We shall call the second term above (56), when properly integrated, *the power radiated from the flat-top.* The third term, since it contains both sets of coördinates, may be called *power radiated mutually.* These designations are merely for convenience in paragraphing the mathematics involved.

15. Power Radiated from the Flat-top. — Let us now enter upon a determination of the power contributed by the second term of the right hand side of equation (56), and integrate this term over the aërial hemisphere; that is, the hemisphere above the surface of the earth regarded as a plane.

The element of area of this hemisphere is

$$dS = r_0^2 \sin \psi \, d\psi \, d\Sigma. \tag{57}$$

This is to be substituted in the required term involving E_ψ and H_Σ; but these quantities involve the coördinate z, which must be replaced by its value in polar coördinates

$$z = r_0 \sin \psi \cos \Sigma. \tag{58}$$

Besides (57) and (58) we are also to substitute the values of E_ψ and H_Σ from (55) into the term

$$dp = \frac{c}{4\pi} \left(E_\psi \, H_\Sigma \right) dS. \tag{59}$$

E_ψ and H_Σ are identical, by (55); the product will give certain terms involving $\sin^2 \tau$, other terms involving $\cos^2 \tau$, and still other terms involving $\sin \tau \cos \tau$; where τ has the value given in (51). If we take the time average for a complete cycle, or, if we prefer, for a time that is large in comparison with a complete period, we have

$$\text{av. } \sin^2 \tau = \quad \text{av. } \cos^2 \tau = \tfrac{1}{2};$$

while the average of the product

$$\text{av. } \sin \tau \cos \tau = 0.$$

The integral form of (59) then becomes, if $\bar{p}$ = the time average of radiated power,

$$\bar{p} = \frac{I^2}{2\pi c} \int_o^{\pi} \frac{d\psi}{\sin\psi} \left[\left\{ \cos^2 B + \cos^2 \psi \sin^2 B + 1 - 2 \cos B \cos (B \cos \psi) \right. \right.$$

$$\left. \left. - 2 \cos \psi \sin B \sin (B \cos \psi) \int_{-\pi/2}^{+\pi/2} d\Sigma \left[\sin^2 (A \sin \psi \cos \Sigma) \right] \right\} \right]. \quad (60)$$

We shall first perform the integration with respect to Σ

$$\int_{-\pi/2}^{\pi/2} d\Sigma \left\{ \sin^2 (A \sin \psi \cos \Sigma) \right\} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\Sigma \left\{ \frac{1 - \cos (2 A \sin \psi \cos \Sigma)}{2} \right\}$$

$$= \frac{\pi}{2} - \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos (2 A \sin \psi \cos \Sigma) \, d\Sigma.$$

$$= \frac{\pi}{2} - \frac{1}{2} \int_{-\frac{\pi}{2}}^{o} \cos (2 A \sin \psi \cos \Sigma) \, d\Sigma - \frac{1}{2} \int_o^{\pi/2} \cos (2 A \sin \psi \cos \Sigma) d\Sigma \quad (61)$$

$$= \frac{\pi}{2} - \frac{1}{2} \int_o^{\pi} \cos (2 A \sin \psi \cos \Sigma) \, d\Sigma. \quad (62)$$

This last step consists in changing the variable of the first integral of the right-hand side of (61) by putting

$$\Sigma' = \pi + \Sigma,$$

which makes the limits $\frac{\pi}{2}$ and π without any other change, except the change of Σ to Σ'.   But since this is the variable of integration, the prime may be omitted, and the terms of (61) added, giving (62).

Equation (62) may now be integrated by Formula (11) Art. 121 of Byerly's *Fourier's Series and Spherical Harmonics* giving for the integral of (62)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\Sigma \; \{\sin^2 (A \sin \psi \cos \Sigma)\} = \frac{\pi}{2} - \frac{\pi}{2} J_0 (2 A \sin \psi), \qquad (63)$$

where J_0 is the Bessel's Function of the zeroth order, with a development of the form

$$J_0 (x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \, 4^2} - \frac{x^6}{2^2 \, 4^2 \, 6^2} + \cdots \cdot \qquad (64)$$

Before substituting in (60) let us simplify the general trigonometric factor in the brace of (60) by placing $\cos^2\psi$ by $1 - \sin^2\psi$, and letting $k = 2A$, as in (42), we then obtain

$$\bar{p} = \frac{I^2}{4c} \int_o^\pi \left\{ \frac{1 - J_0 (k \sin \psi)}{\sin \psi} \right\} \left\{ 2 - \sin^2 \psi \sin^2 B \right.$$

$$\left. - 2 \cos B \cos (B \cos \psi) - 2 \cos \psi \sin B \sin (B \cos \psi) \right\} d\psi$$

$$= \frac{I^2}{4c} \int_o^\pi \left\{ \frac{k^2 \sin^2 \psi}{2^2} - \frac{k^4 \sin^3 \psi}{2^2 \, 4^2} + \frac{k^6 \sin^5 \psi}{2^2 \, 4^2 \, 6^2} - \cdots \right\}$$

$$\left\{ 2 - \sin^2 \psi \sin^2 B - 2 \cos B \cos (B \cos \psi) \right.$$

$$\left. - 2 \cos \psi \sin B \sin (B \cos \psi) \right\} d\psi. \qquad (65)$$

or

$$\bar{p} = \frac{I^2}{4c} \left[- 2 \sum (-1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \cdots n^2} \int_o^\pi \sin^{n-1} \psi d\psi \right.$$

$$+ \sin^2 B \sum (-1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \cdots n^2} \int_o^\pi \sin^{n+1} \psi d\psi$$

$$+ 2 \cos B \sum (-1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \cdots n^2} \int_o^\pi \sin^{n-1} \psi \cos (B \cos \psi) \, d\psi$$

$$+ 2 \sin B \sum (-1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \cdots n^2} \int_o^\pi \sin^{n-1} \psi \cos \psi \sin$$

$$\left. (B \cos \psi) \, d\psi \right]$$

$$n = 2, \; 4, \; 6, \ldots \ldots \qquad (66)$$

Treating these several integrals separately, we have

$$\int_0^\pi \sin^{n-1} \psi d\psi = \int_0^{\frac{\pi}{2}} \sin^{n-1} \psi d\psi + \int_{\frac{\pi}{2}}^\pi \sin^{n-1} \psi d\psi$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-1} \psi d\psi + \int_0^{\frac{\pi}{2}} \cos^{n-1} \psi d\psi$$

$$= 2 \left\{ \frac{2 \cdot 4 \cdot 6 \cdots n - 2}{1 \cdot 3 \cdot 5 \cdots n - 1} \right\} \tag{67}$$

by B. O. Peirce's Tables, Formula No. 483.
Likewise

$$\int_0^\pi \sin^{n+1} \psi d\psi = 2 \left\{ \frac{2 \cdot 4 \cdot 6 \cdots n}{1 \cdot 3 \cdot 5 \cdots n + 1} \right\}. \tag{68}$$

Now by Byerly's *Fourier's Series and Spherical Harmonics* Equation (9), Art. 122,

$$\int_0^\pi \sin^{n-1} \psi \, \cos \left(B \cos \psi \right) d\psi = \frac{2^{\frac{n-1}{2}} \sqrt{\pi} \, \Gamma \left(\frac{n}{2} \right)}{B^{\frac{n-1}{2}}} J_{\frac{n-1}{2}} (B) \tag{69}$$

where $J_{\frac{n-1}{2}} (B)$ is a Bessel's Function of the order $(n-1)/2$, and $\Gamma \left(\frac{n}{2} \right)$ is the Gamma Function of $\frac{n}{2}$.

For the last integral of (66), we have by Problem 2 and Equation (9) of the same article of Byerly's *Fourier's Series*

$$\int_0^\pi \sin^{n-1} \psi \, \cos \psi \, \sin \left(B \cos \psi \right) d\psi$$

$$= \frac{B}{n} \int_0^\pi \sin^{n+1} \psi \, \cos \left(B \cos \psi \right) d\psi$$

$$= \frac{B}{n} \cdot \frac{2^{\frac{n+1}{2}} \sqrt{\pi} \, \Gamma \left(\frac{n}{2} + 1 \right)}{B^{\frac{n+1}{2}}} J_{\frac{n+1}{2}} (B). \tag{70}$$

Substituting these various integrations (67), (68), (69), and (70) in (66), we have

$$\bar{p} = \frac{1^2}{4c}\left[-4\sum(-1)^{\frac{n}{2}}\frac{k^n}{n!n} + 2\sin^2 B\sum(-1)^{\frac{n}{2}}\frac{k^n}{n+1!}\right.$$

$$+ 2\cos B\sum(-1)^{\frac{n}{2}}\frac{k^n}{2^2\,4^2\cdots n^2}\;\frac{2^{\frac{n-1}{2}}\sqrt{\pi}\,\Gamma\left(\dfrac{n}{2}\right)}{B^{\frac{n-1}{2}}}J_{\frac{n-1}{2}}(B)$$

$$+ 2\sin B\sum(-1)^{\frac{n}{2}}\frac{k^n}{2^2 4^2\cdots n^2}\frac{B}{n}\;\frac{2^{\frac{n+1}{2}}\sqrt{\pi}\,\Gamma\left(\dfrac{n}{2}+1\right)}{B^{\frac{n+1}{2}}}J_{\frac{n+1}{2}}(B)\cdot$$

$$\tag{71}$$

$$n = 2, 4, 6, \cdots \infty \quad . \qquad B = \frac{2\pi b}{\lambda} \text{ is between 0 and } \frac{\pi}{2}.$$

This result may be expessed in a power series by expanding the Bessel's Functions by equation (6), Art. 120 of Byerly's *Fourier's Series*, giving

$$J_{\frac{n-1}{2}}(B) = \frac{B^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}}\Gamma\left(\dfrac{n+1}{2}\right)}\left[1 - \frac{B^2}{2^2\left(\dfrac{n+1}{2}\right)}\right.$$

$$+ \frac{B^4}{2!2^4\left(\dfrac{n+1}{2}\right)\left(\dfrac{n+3}{2}\right)} - \frac{B^6}{3!2^6\left(\dfrac{n+1}{2}\right)\left(\dfrac{n+3}{2}\right)\left(\dfrac{n+5}{2}\right)} + \cdots\right]$$

$$\tag{72}$$

$$J_{\frac{n+1}{2}}(B) = \frac{B^{\frac{n+1}{2}}}{2^{\frac{n+1}{2}}\Gamma\left(\dfrac{n+3}{2}\right)}\left[1 - \frac{B^2}{2^2\left(\dfrac{n+3}{2}\right)}\right.$$

$$+ \frac{B^4}{2!2^4\left(\dfrac{n+3}{2}\right)\left(\dfrac{n+5}{2}\right)} - \frac{B^6}{3!2^6\left(\dfrac{n+3}{2}\right)\left(\dfrac{n+5}{2}\right)\left(\dfrac{n+7}{2}\right)} + \cdots\right].$$

$$\tag{73}$$

Note that

$$\sqrt{\pi}\,\frac{\Gamma\left(\dfrac{n}{2}\right)}{\Gamma\left(\dfrac{n+1}{2}\right)} = 2\frac{2\cdot4\cdot6\cdots n-2}{1\cdot3\cdot5\cdots n-1}\;, \tag{74}$$

and

$$\sqrt{\pi}\,\frac{\Gamma\left(\dfrac{n}{2}+1\right)}{\Gamma\left(\dfrac{n+3}{2}\right)} = 2\frac{2\cdot4\cdot6\cdots n}{1\cdot3\cdot5\cdots n+1}. \tag{75}$$

Putting these values in equation (71) we obtain

$$\begin{aligned}
\bar{p} = \frac{I^2}{c}\sum(-1)^{\frac{n}{2}}\frac{k^n}{n!}&\left[-\frac{1}{n}+\frac{\sin^2 B}{2\,(n+1)}\right.\\
&+\cos B\,\frac{1}{n}\left\{1-\frac{B^2}{2\,(n+1)}+\frac{B^4}{2!2^2}\frac{1}{(n+1)\,(n+3)}\right.\\
&\qquad\left.-\frac{B^6}{3!2^3}\frac{1}{(n+1)\,(n+3)\,(n+5)}+\cdots\cdots\right\}\\
&+B\sin B\,\frac{1}{n}\left\{\frac{1}{n+1}-\frac{B^2}{2}\frac{1}{(n+1)\,(n+3)}\right.\\
&\qquad+\frac{B^4}{2^2 2}\frac{1}{(n+1)\,(n+3)\,(n+5)}\\
&\qquad\left.\left.-\frac{B^6}{2^3 3}\frac{1}{(n+1)\,(n+3)\,(n+5)\,(n+7)}+\cdots\cdots\right\}\right],
\end{aligned}$$

where

$$n = 2, 4, 6, \ldots. \tag{76}$$

Equation (76) may be further improved for purposes of calculation by expanding the trigonometric functions in power series and collecting the terms. For this purpose

$$\frac{\sin^2 B}{2} = \frac{1-\cos 2B}{4} = \frac{B^2}{2!} - \frac{2^2 B^4}{4!} + \frac{2^4 B^6}{6!} - \frac{2^6 B^8}{8!} + \cdots\cdots, \tag{77}$$

$$\cos B = 1 - \frac{B^2}{2!} + \frac{B^4}{4!} - \frac{B^6}{6!} + \cdots\cdots, \tag{78}$$

$$B \sin B = B^2 - \frac{B^4}{3!} + \frac{B^6}{5!} - \cdots\cdots \tag{79}$$

Equations (77), (78) and (79) substituted in (76) will give

$$\bar{p} = \frac{I^2}{c} \sum (-1)^{\frac{n}{2}} \frac{k^n}{n!} \cdot F_n(B), \tag{80}$$

where $F_n(B)$ is a polynomial in B^0, B^2, B^4, etc., where the coefficients of the several powers of B are contained in the table of page 225.

In this table the bottom row of terms gives the coefficients of the powers of B, when the summation indicated in (80) is performed with $n = 2, 4, 6 \cdots \infty$. The various terms in the columns were employed in obtaining the last row by addition.

The coefficient of B^{10} is not contained in the table, because of its numerous terms, but its value when summed up is

$$\frac{255n^4 + 6084n^3 + 51396n^2 + 177264n + 193536}{10!\,(n+1)\,(n+3)\,(n+5)\,(n+7)\,(n+9)}$$

Substituting the values of the coefficients multiplied by the corresponding powers of B and summing up as indicated in equation (80), we obtain for the power the expression

$$
\begin{aligned}
p = \frac{I^2}{c} \Bigg[& k^2 \left\{ \frac{B^4}{60} - \frac{11B^6}{3780} + \frac{13B^8}{56700} - \frac{B^{10}}{93555} + \cdots\cdots \right\} \\
& - k^4 \left\{ \frac{B^4}{1120} - \frac{B^6}{6480} + \frac{B^8}{83160} - \frac{B^{10}}{77395500} + \cdots\cdots \right\} \\
& + k^6 \left\{ \frac{B^4}{45360} - \frac{B^6}{24960960} + \frac{7B^8}{6!34720} - \cdots\cdots \right\}
\end{aligned} \tag{81}
$$

This equation gives the average power radiated in the aërial hemisphere from the flat-top of the antenna regarded as a separate radiator with the distribution that it has under the fundamental assumptions of the problem. The current is to be measured in absolute electrostatic units, and the power is in ergs per second.

B^0	B^2	B^4	B^6	B^8
$-\dfrac{1}{n}$	$+\dfrac{1}{2!(n+1)}$	$-\dfrac{2^2}{4!(n+1)}$	$+\dfrac{2^4}{6!(n+1)}$	$-\dfrac{2^6}{8!(n+1)}$
$+\dfrac{1}{n}$	$-\dfrac{1}{2n(n+1)}$	$+\dfrac{1}{2!2^2 n(n+1)(n+3)}$	$-\dfrac{1}{3!2^3 n(n+1)(n+3)(n+5)}$	$+\dfrac{1}{4!2^4 n(n+1)(n+3)(n+5)(n+7)}$
	$-\dfrac{1}{2!n}$	$+\dfrac{1}{2!2^2 n(n+1)}$	$-\dfrac{1}{2^2 2!2! n(n+1)(n+3)}$	$+\dfrac{1}{2!3!2^3 n(n+1)(n+3)(n+5)}$
	$+\dfrac{1}{n(n+1)}$	$+\dfrac{1}{4!n}$	$-\dfrac{1}{4!2 n(n+1)}$	$+\dfrac{1}{2!2^2 4! n(n+1)(n+3)}$
		$-\dfrac{1}{2n(n+1)(n+3)}$	$-\dfrac{1}{6!n}$	$+\dfrac{1}{6!2 n(n+1)}$
		$-\dfrac{1}{3!n(n+1)}$	$+\dfrac{1}{2!2^2 n(n+1)(n+3)(n+5)}$	$+\dfrac{1}{8!n}$
			$+\dfrac{1}{3!2 n(n+1)(n+3)}$	$-\dfrac{1}{3!2^3 n(n+1)(n+3)(n+5)(n+7)}$
			$+\dfrac{1}{5!n(n+1)}$	$-\dfrac{1}{3!2!2^2 n(n+1)(n+3)(n+5)}$
				$-\dfrac{1}{5!2 n(n+1)(n+3)}$
				$-\dfrac{1}{7!n(n+1)}$
0	0	$-\dfrac{n+2}{4\cdot2(n+1)(n+3)}$	$+\dfrac{3n^2+22n+32}{4!6(n+1)(n+3)(n+5)}$	$-\dfrac{3n^3+44n^2+196n+240}{5!2(n+1)(n+3)(n+5)(n+7)}$

In this equation
$$B = \frac{2\pi b}{\lambda}$$

$$k = 2A = \frac{4\pi a}{\lambda}.$$

It remains to find how this power is modified by the mutual effect consisting of the interference between the waves emitted from the vertical portion of the antenna and the waves emitted from the horizontal part. This is the subject matter of Part III.

PART III.

The Mutual Term in Power Determination.

16. The Trigonometric Relations. — In Section **14,** equation (56), it has been shown that the power radiated through an element of surface consists of three terms in the form

$$dp = \frac{c}{4\pi} \left(E_\theta H_\phi + E_\psi H_\Sigma + 2 \cos \alpha \, E_\theta H_\psi \right) dS.$$

The first two of these terms we have already discussed. Putting in the values of E_θ and H_ψ from equations (19) and (55) the remaining power term, which we have for convenience called *mutual power,* becomes in the time average

$$d\overline{p} = \frac{I^2 \cos \alpha \, dS}{\pi c \, r^2{}_0 \sin \theta \sin \psi} \sin \frac{Az}{r_0} \left\{ \cos \psi \sin B - \sin (B \cos \psi) \right\}$$

$$\left\{ \cos B \cos (A \cos \theta) - \sin B \cos \theta \sin (A \cos \theta) - \cos G \right\}. \quad (82)$$

In forming this equation we have multiplied the expression for E_θ of eq. (19) by the expression for H_ψ, eq. (55).   The product so obtained contains terms involving $sin\, \tau \cos \tau$ plus terms involving $\cos^2\tau$.   The time average of the $\sin \tau \cos \tau$ terms is zero; while the time average of $\cos^2 \tau$ is $\frac{1}{2}$; these facts have been used in forming (82).

To be able to integrate equation (82) we must replace a, z, ψ and dS by their values in terms of θ, ϕ and r_0.   By Fig. 3,

$$z = r_0 \cos \theta, \quad (83)$$

$$dS = r_0{}^2 \sin \theta \, d\theta \, d\phi. \quad (84)$$

In the spherical triangle of Figure 10, α is represented, as defined, as the angle between θ and ψ, while opposite to α the side is $\pi/2$.   The important trigonometric relation in a spherical triangle is as follows:

I.   The cosine of any side is equal to the product of the cosines of the two other sides plus the continued product of the sines of these sides and the cosine of the included angle.

By this proposition, referring to Figure 10, we see that

$$\cos \psi = \cos \frac{\pi}{2} \cos \theta + \sin \frac{\pi}{2} \sin \theta \cos \phi$$
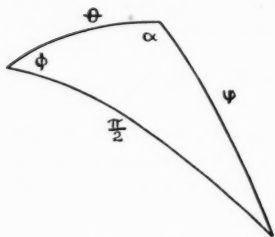
$$= \sin \ \theta \cos \phi. \tag{85}$$

FIGURE 10.

By the same proposition

$$\cos \frac{\pi}{2} = \cos \theta \cos \psi + \sin \theta \sin \psi \cos a;$$

∴

$$\cos a = - \frac{\cos \theta \cos \psi}{\sin \theta \sin^2 \psi}, \tag{86}$$

or

$$\frac{\cos a}{\sin \psi} = - \frac{\cos \theta \cos \psi}{\sin \theta \sin^2 \psi}; \tag{87}$$

and by (85) this becomes

$$\frac{\cos a}{\sin \psi} = - \frac{\cos \theta \cos \phi}{1 - \sin^2 \theta \cos^2 \phi}. \tag{88}$$

17.   Integration for Mutual Power. — Now substituting the trigonometric relations (83), (84), (85), (88) into equation (82), we obtain the following integral expression for the time average of the mutual power radiated through the *aërial hemisphere*:

$$\bar{p} = \frac{I^2}{c\pi} \int_o^{\pi/2} d\theta \sin(A \cos \theta) \left\{ \cos B \cos(A \cos \theta) - \right.$$

$$\left. \sin B \cos \theta \sin(A \cos \theta) - \cos G \right\}$$

$$\left[\cos \theta \int_o^{2\pi} \frac{\cos \phi \sin(B \sin \theta \cos \phi)\, d\phi}{1 - \sin^2 \theta \cos^2 \phi} \right.$$

$$\left. - \cos \theta \sin \theta \sin B \int_o^{2\pi} \frac{\cos \phi\, d\phi}{1 - \sin^2 \theta \cos^2 \phi} \right]. \tag{89}$$

This is a very complicated expression involving the integral of an integral.

We shall first proceed to perform the integration with respect to ϕ.

$$\text{Let } V = \int_o^{2\pi} \frac{\cos \phi \sin(B \sin \theta \cos \phi)\, d\phi}{1 - \sin^2 \theta \cos^2 \phi} \tag{90}$$

and break the integral into the sum of two integrals thus:

$$V = \int_o^{\pi} + \int_{\pi}^{2\pi}.$$

By a change of variable in the second of these two integrals by replacing ϕ by $\phi' + \pi$, we find that the integrand is unchanged, while the limits become 0 and π, so we may write

$$V = 2 \int_o^{\pi} \frac{\cos \phi \sin(B \sin \theta \cos \phi)\, d\phi}{1 - \sin^2 \theta \cos^2 \phi}. \tag{91}$$

Again decomposing this into the sum of two integrals we have

$$V = 2 \left\{ \int_o^{\pi/2} + \int_{\pi/2}^{\pi} \right\} \tag{92}$$

and changing the variable in the second integral by putting $\phi = \pi - \phi'$, the second integral becomes

$$\int_{\pi/2}^{\pi} = \int_{\pi/2}^{o} \frac{-d\phi'(-\cos \phi')(-\sin(B \sin \theta \cos \phi'))}{1 - \sin^2 \theta \cos^2 \phi'},$$

which by dropping the primes and substituting in (92) and (91) gives

$$V = 4 \int_o^{\frac{\pi}{2}} \frac{\cos \phi \sin (B \sin \theta \cos \phi) \, d\phi}{1 - \sin^2 \theta \cos^2 \phi}. \tag{93}$$

Now expanding in series as follows:

$$\sin (B \sin \theta \cos \phi) = B \sin \theta \cos \phi - \frac{B^3 \sin^3 \theta \cos^3 \phi}{3!} + \frac{B^5 \sin^5 \theta \cos^5 \phi}{5!} - \cdots,$$

and

$$\frac{1}{1 - \sin^2 \theta \cos^2 \phi} = 1 + \sin^2 \theta \cos^2 \phi + \sin^4 \theta \cos^4 \phi + \ldots; \tag{93a}$$

and by taking the product of these two series we obtain

$$\begin{aligned} V = 4 \int_o^{\pi/2} d\phi \bigg[& B \sin \theta \cos^2 \phi \\ & + \left\{ B - \frac{B^3}{3!} \right\} \sin^3 \theta \cos^4 \phi \\ & + \left\{ B - \frac{B^3}{3!} + \frac{B^5}{5!} \right\} \sin^5 \theta \cos^6 \phi \\ & + \cdots \cdots \bigg]. \end{aligned} \tag{94}$$

Integrating (94) by formula 483 of B. O. Peirce's Tables, we obtain

$$\begin{aligned} V = 2\pi \bigg[& \frac{1}{2} B \sin \theta \\ & + \frac{1 \cdot 3}{2 \cdot 4} \left\{ B - \frac{B^3}{3!} \right\} \sin^3 \theta \\ & + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left\{ B - \frac{B^3}{3!} + \frac{B^5}{5!} \right\} \sin^5 \theta \\ & + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \left\{ B - \frac{B^3}{3!} + \frac{B^5}{5!} - \frac{B^7}{7!} \right\} \sin^7 \theta \\ & \cdots \cdots \cdots \end{aligned} \tag{95}$$

We shall next proceed to perform the second integration with respect to ϕ indicated in (89). For abbreviation let us write

$$W = \int_o^{2\pi} \frac{\cos^2 \phi \, d\phi}{1 - \sin^2 \theta \cos^2 \phi} = 4 \int_o^{\pi/2} \frac{\cos^2 \phi \, d\phi}{1 - \sin^2 \theta \cos^2 \phi}$$

by reasoning similar to the above. Expanding the denominator by (93a), we have

$$W = 4 \int_o^{\pi/2} d\phi \cos^2 \phi \left\{ 1 + \sin^2 \theta \cos^2 \phi + \sin^4 \theta \cos^4 \phi + \cdots \right\}$$

$$= 2\pi \left\{ \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} \sin^2 \theta + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \sin^4 \theta + \text{-------} \right\} \cdot \tag{96}$$

(If we need it, this integral can be obtained by direct integration in the form

$$W = 2\pi \left\{ \frac{1}{\cos \theta \, (1 + \cos \theta)} \right\}$$

but the expanded form is more useful for our purpose).

Now substituting (95) and (96) in (89) we obtain

$$\bar{p} = \frac{2 \, I^2}{c} \int_o^{\pi/2} d\theta \sin \theta \sin (A \cos \theta) \left\{ \cos B \cos (A \cos \theta) \right.$$

$$- \sin B \cos \theta \sin (A \cos \theta) - \cos G \left. \right\}$$

$$\left[\frac{1}{2} (B - \sin B) \sin \theta \right.$$

$$+ \frac{1 \cdot 3}{2 \cdot 4} \left(B - \frac{B^3}{3!} - \sin B \right) \sin^3 \theta$$

$$+ \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B \right) \sin^5 \theta$$

$$+ \cdots\cdots\cdots \left. \right] \cdot \tag{97}$$

To evaluate this expression we must obtain the following integrals:

$$I_1 = \int_o^{\frac{\pi}{2}} d\theta \sin^n \theta \, \frac{\sin (2A \cos \theta)}{2}, \tag{98}$$

$$I_2 = \int_o^{\pi/2} d\theta \sin^n \theta \cos \theta \sin^2 (A \cos \theta), \tag{99}$$

$$I_3 = \int_o^{\pi/2} d\theta \sin^n \theta \sin (A \cos \theta), \tag{100}$$

where $n = 2, 4, 6, 8, \ldots \ldots$

I_3 is the simplest of these integrals and will be considered first.   By expanding $\sin (A \cos \theta)$ in series we have

$$I_3 = \int_o^{\pi/2} d\theta \sin^n \theta \left\{ A \cos \theta - \frac{A^3 \cos^3 \theta}{3!} + \frac{A^5 \cos^5 \theta}{5!} - \cdots \right\}$$

which by Byerly Int. Calc., Art. 99, Ex. 2, may be integrated in Gamma Functions as follows:

$$I_3 = \frac{A}{n+1} - \frac{A^3}{3!} \frac{\Gamma(2) \Gamma\left(\dfrac{n+1}{2}\right)}{2 \Gamma\left(\dfrac{n+3}{2} + 1\right)}$$

$$+ \frac{A^5}{5!} \frac{\Gamma(3) \Gamma\left(\dfrac{n+1}{2}\right)}{2 \Gamma\left(\dfrac{n+5}{2} + 1\right)}$$

$$- \frac{A^7}{7!} \frac{\Gamma(4) \Gamma\left(\dfrac{n+1}{2}\right)}{2 \Gamma\left(\dfrac{n+7}{2} + 1\right)}$$

$$+ \quad \cdots \cdots \cdots \tag{101}$$

If we note that

$$\Gamma\left(\frac{n+3}{2} + 1\right) = \frac{n+3}{2} \frac{n+1}{2} \Gamma\left(\frac{n+1}{2}\right)$$

$$\Gamma\left(\frac{n+5}{2} + 1\right) = \frac{n+5}{2} \frac{n+3}{2} \frac{n+1}{2} \Gamma\left(\frac{n+1}{2}\right)$$

$$\Gamma(2) = 1$$
$$\Gamma(3) = 2!$$
$$\Gamma(4) = 3!$$

we obtain

$$I_3 = \frac{A}{n+1} - \frac{A^3}{3!} \frac{2}{(n+1)(n+3)} + \frac{A^5}{5!} \frac{2^2 2!}{(n+1)(n+3)(n+5)} - \cdots$$

$$= \frac{A}{n+1} \left\{ 1 - \frac{A^2}{3(n+3)} + \frac{A^4}{5 \cdot 3(n+3)(n+5)} - \frac{A^6}{7 \cdot 5 \cdot 3(n+3)(n+5)(n+7)} + \cdots \right\} \cdot \quad (102)$$

In like manner

$$I_1 = \frac{2A}{2(n+1)} \left\{ 1 - \frac{(2A)^2}{3(n+3)} + \frac{(2A)^4}{5 \cdot 3(n+3)(n+5)} - \frac{(2A)^6}{7 \cdot 5 \cdot 3(n+3)(n+5)(n+7)} + \cdots \right\} \cdot \quad (103)$$

Now taking up integral I_2 from equation (99), let us write it

$$I_2 = \int_0^{\pi/2} d\phi \, \sin^n \theta \cos \theta \left\{ \frac{1 - \cos(2A \cos \theta)}{2} \right\},$$

and expanding cos (2 A cos θ) in series, obtain

$$I_2 = \frac{1}{2} \int d\phi \left[\sin^n \theta \cos \theta \left\{ \frac{(2A)^2 \cos^2 \theta}{2!} - \frac{(2a)^4 \cos^4 \theta}{4!} + \cdots \cdots \right\} \right].$$

This equation, integrated in Gamma Functions between the limits 0 and $\pi/2$ gives

$$I_2 = \frac{1}{2} \left[\frac{(2A)^2}{2!} \frac{\Gamma\left(\frac{n+1}{2} \Gamma(2)\right)}{2\Gamma\left(\frac{n+3}{2}+1\right)} - \frac{(2A)^4}{4!} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma(3)}{2\Gamma\left(\frac{n+5}{2}+1\right)} + \cdots \cdots \right],$$

$$= \frac{2A^2}{(n+1)(n+3)} \left\{ 1 - \frac{2A^2}{3(n+5)} + \frac{2A^4}{5 \cdot 3(n+5)(n+7)} - \cdots \right\}. \quad (104)$$

Employing the values of I_1, I_2, I_3 found in equations (103), (104) and (102) we may write the expression for the mutual power in the integrated form

$$\bar{p} = \frac{2I^2}{c}\left[\cos B \left\{ \frac{1}{2}(B - \sin B)\frac{A}{3}\left[1 - \frac{(2A)^2}{3\cdot5} + \frac{(2A)^4}{5\cdot3\cdot5\cdot7} - \right.\right.\right.$$
$$\left.\frac{(2A)^6}{7\cdot5\cdot3\cdot5\cdot7\cdot9} + \cdots\right]$$

$$+ \frac{1\cdot3}{2\cdot4}\left(B - \frac{B^3}{3!} - \sin B\right)\frac{A}{5}\left[1 - \frac{(2A)^2}{3\cdot7} + \frac{(2A)^4}{5\cdot3\cdot7\cdot9} - \right.$$
$$\left.\frac{(2A)^6}{7\cdot5\cdot3\cdot7\cdot9\cdot11} + \cdots\right]$$

$$+ \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B\right)\frac{A}{7}\left[1 - \frac{(2A)^2}{3\cdot9} + \frac{(2A)^4}{5\cdot3\cdot9\cdot11} - \right.$$
$$\left.\frac{(2A)^6}{7\cdot5\cdot3\cdot9\cdot11\cdot13} + \cdots\right]$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots \Bigg\}$$

$$- \sin B \left\{ \frac{1}{2}\left(B - \sin B\right)\frac{2A^2}{3\cdot5}\left[1 - \frac{(2A)^2}{3\cdot7} + \frac{(2A)^4}{5\cdot3\cdot7\cdot9} - \right.\right.$$
$$\left.\frac{(2A)^6}{7\cdot5\cdot3\cdot7\cdot9\cdot11} + \cdots\right]$$

$$+ \frac{1\cdot3}{2\cdot4}\left(\left(B - \frac{B^3}{3!} - \sin B\right)\frac{2A^2}{5\cdot7}\left[1 - \frac{(2A)^2}{3\cdot9} + \frac{(2A)^4}{5\cdot3\cdot9\cdot11} - \right.\right.$$
$$\left.\frac{(2A)^6}{7\cdot5\cdot3\cdot9\cdot11\cdot13} + \cdots\right]$$

$$+ \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B\right)\frac{2A^2}{7\cdot9}\left[1 - \frac{(2A)^2}{3\cdot11} + \frac{(2A)^4}{5\cdot3\cdot11\cdot13} - \right.$$
$$\left.\frac{(2A)^6}{7\cdot5\cdot3\cdot11\cdot13\cdot15} + \cdots\right]$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots \Bigg\}$$

$$- \cos G \left\{ \frac{1}{2}\left(B - \sin B\right)\frac{A}{3}\left[1 - \frac{A^2}{3\cdot5} + \frac{A^4}{5\cdot3\cdot5\cdot7} - \right.\right.$$
$$\left.\frac{A^6}{7\cdot5\cdot3\cdot5\cdot7\cdot9} + \cdots\right]$$

$$+ \frac{1\cdot3}{2\cdot4}\left(B - \frac{B^3}{3!} - \sin B\right)\frac{A}{5}\left[1 - \frac{A^2}{3\cdot7} + \frac{A^4}{5\cdot3\cdot7\cdot9} - \right.$$
$$\left.\frac{A^6}{7\cdot5\cdot3\cdot7\cdot9\cdot11} + \cdots\right]$$

$$+ \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B\right)\frac{A}{7}\left[1 - \frac{A^2}{3\cdot9} + \frac{A^4}{5\cdot3\cdot9\cdot11} - \right.$$
$$\left.\frac{A^6}{7\cdot5\cdot3\cdot9\cdot11\cdot13} + \cdots\right]$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots \Bigg\} \Bigg]. \tag{104}$$

If now we recall that $G = A + B$, it will be seen that the equation (104) is entirely in terms of A and B and I.

For purpose of computation it is found advisable to expand all of the trigonometrical expressions in power series and then perform with them the indicated operations.   This was done with considerable labor and gave the following expression for mutual power:

$$\bar{p} = \frac{2I^2}{c}\Bigg[A^2 \Big\{ \ .0166\,B^4 - .00404\,B^6 + .000390\,B^8 - $$
$$.0000144\,B^{10} + \cdots \Big\{$$
$$+ A^3 \Big\{ \ .0083\,B^3 - .00480\,B^5 + .000729\,B^7 - $$
$$.0000486\,B^9 + \cdots \Big\{$$
$$+ A^4 \Big\{ - .00433\,B^4 + .00104\,B^6 - .000102\,B^8 + $$
$$.0000051\,B^{10} - \cdots \Big\{$$
$$+ A^5 \Big\{ - .00127\,B^3 + .000741\,B^5 - .000106\,B^7 + $$
$$.0000073\,B^9 - \cdots \Big\{$$
$$+ A^6 \Big\{ .000404\,B^4 - .000101\,B^6 + .0000101\,B^8 - $$
$$.0000005\,B^{10} + $$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \Bigg]. \tag{105}$$

This equation gives the time average of the power radiated in the aërial hemisphere by the mutual effect of the fields from both parts of the antenna and is the correction to be added to the power radiated by the two parts, estimated as independent of one another.   The current I is in absolute c.g.s. electrostatic units, and the power is in ergs per second.

18.   Summation of Flat-top Power and Mutual Power. —

We have obtained in equation (81) the time average of flat-top radiated power, and in equation (105) the time average of mutual radiated

power.   If we replace the k of (81) by its value in terms of A, the two expressions may be added together.   At the time of the addition we shall reduce the units to the practical system by multiplying the right hand sides of both power equations by 30 times the velocity of light in centimeters per second (i. e. by 30 c), and obtain

$$\bar{p} = 60 \, I^2 \left[A^2 \left\{ .05 \quad B^4 - .00985 \; B^6 + .000849 \; B^8 - \right. \right.$$

$$\left. .0000356 \; B^{10} \cdots \right\}$$

$$+ A^3 \left\{ .0083 \; B^3 - .00480 \; B^5 + .000729 \; B^7 - \right.$$

$$\left. .0000486 \; B^9 + .. \right\}$$

$$- A^4 \left\{ .01148 \; B^4 - .00227 \; B^6 + .000198 \; B^8 - \right.$$

$$\left. .00000953 \; B^{10} + \ldots \right\}$$

$$- A^5 \left\{ .00127 \; B^3 - .000741 \; B^5 + .000106 \; B^7 - \right.$$

$$\left. .0000073 \; B^9 + \ldots \right\}$$

$$+ A^6 \left\{ .00111 \; B^4 - .00014 \; B^6 + .000019 \; B^8 - \ldots \ldots \right\}$$

$$\left. + \ldots \ldots \right].$$

$$(106)$$

This is the total power contribution of the flat top by virtue of its individual and mutual action.   The power is in watts, and the current I is in amperes.

Certain Tables computed in the next Part of this communication make calculations with this series comparatively simple.

PART IV.

COMPUTATIONS OF RADIATION RESISTANCE.

19. Equation for Radiation Resistance. — If

a = length of vertical part in meters,
b = length of horizontal part in meters,
λ_0 = the natural wavelength of the antenna in meters,
λ = the wavelength in meters of the antenna as loaded with inductance
    at its base,

$$A = \frac{2\pi a}{\lambda},$$

$$B = \frac{2\pi b}{\lambda},$$

$$q = \frac{\pi \lambda_0}{\lambda},$$

we may obtain the radiation resistance of the antenna by dividing the power radiated by the mean square of the current at the base of the antenna. This mean square current at the base of the antenna is by (5)

$$I_0^2 = \frac{I^2 \sin^2 (q/2)}{2}$$

Performing this division as to the flat-top power employing equation (106) and adding the result to the radiation resistance for the vertical portion as given in equation (44) we obtain for the total radiation resistance of the antenna the equation

$$R_\Omega = \frac{1}{\sin^2 (q/2)} \left\{ R_1 - R_2 \cos q - R_3 \sin q + \right.$$
$$\left. r_2 A^2 + r_3 A^3 - r_4 A^4 - r_5 A^5 + r_6 A^6 + \cdots \right\} \cdot \quad (107)$$

This is Radiation Resistance in Ohms, where

$$R_1 = 15 \left\{ \frac{2+2}{3!2} (2A)^2 - \frac{4+2}{5!4} (2A)^4 + \frac{6+2}{7!6} (2A)^6 - \cdots \right\}$$

$$R_2 = 15 \left\{ \frac{2^2 + 2^2 - 4}{3!2} (2A)^2 - \frac{4^2 + 2^4 - 6}{5!4} (2A)^4 + \right.$$
$$\left. \frac{6^2 + 2^6 - 8}{7!6} (2A)^6 - \cdots \right\}$$

$$R_3 = 15 \left\{ \frac{3^2 + 2^3 - 5}{4!3} (2A)^3 - \frac{5^2 + 2^5 - 7}{6!5} (2A)^5 + \right.$$
$$\left. \frac{7^2 + 2^7 - 9}{8!7} (2A)^7 - \cdots \right\}$$

$$r_2 = 120 \left\{ .05\, B^4 - .00985\, B^6 + .000849\, B^8 - .0000356\, B^{10} + \cdots \right\}$$

$$r_3 = 120 \left\{ .0083\, B^3 - .00480\, B^5 + .000729\, B^7 - \right.$$
$$\left. .0000486\, B^9 + \cdots \right\}$$

$$r_4 = 120 \left\{ .01148\, B^4 - .00227\, B^6 + .000198\, B^8 - \right.$$
$$\left. .00000953\, B^{10} + \cdots \right\}$$

$$r_5 = 120 \left\{ .00127\, B^3 - .000741\, B^5 + .000106\, B^7 - \right.$$
$$\left. .0000073\, B^9 + \cdots \right\}$$

$$r_6 = 120 \left\{ .00111\, B^4 - .00014\, B^6 + .000019\, B^8 - \cdots \right\} . \tag{108}$$

20. Tables of Coefficients of Radiation Resistance. — There follow in Tables I and II the values of the coefficients R_1, R_2, R_3, r_2, r_3, r_4, r_5, r_6 for various values of A and B respectively. These tables have been computed by the equations (108).

TABLE I.

COEFFICIENTS R_1, R_2, AND R_3.

$2A$	$\lambda/4a$	R_1	R_2	R_3
.1	31.416	.04998	.049919	.002498
.2	15.70	.19971	.19870	.01994
.3	10.47	.44848	.44344	.06700
.4	7.85	.79521	.78107	.1579
.5	6.28	1.2383	1.20634	.3060
.6	5.236	1.7759	1.6969	.5241
.7	4.488	2.4055	2.2602	.8232
.8	3.927	3.1240	2.8786	1.2137
.9	3.491	3.9290	3.5403	1.696
1.0	3.141	4.8165	4.2315	2.300
1.1	2.854	5.7837	4.9383	3.009
1.2	2.616	6.8232	5.6442	3.823
1.4	2.241	9.150	7.000	5.90
1.5	2.092	10.3392	7.611	6.999
1.6	1.962	11.64	8.15	8.35
1.732	1.812	13.415	8.798	10.113
1.8	1.743	14.40	9.10	11.20
2.00	1.570	17.241	9.550	14.354
2.20	1.427	20.15	9.55	17.80
2.236	1.403	20.778	9.508	18.470
2.40	1.307	23.22	9.00	21.42
2.60	1.207	26.37	7.90	25.20
2.642	1.189	27.053	7.60	25.927
2.80	1.121	29.40	6.22	29.05
3.141	1.000	34.45	2.12	35.64

TABLE II.

COEFFICIENTS r_2, r_3, ETC.

B	$\lambda/4b$	r_2	r_3	r_4	r_5	r_6
1.4	1.112	15.8	.686	3.55	.055	.481
1.2	1.31	9.32	.585	2.13	.107	.234
1.0	1.57	4.92	.498	1.13	.075	.118
.8	1.96	2.16	.340	.513	.051	.050
.6	2.61	.73	.171	.158	.026	.016
.4	3.93	.15	.057	.035	.009	.004
.2	7.85	.009	.008	.002	.001	.0002

**21. Curves of Resistance Due to Radiation from the Flat-
top.** — We shall now proceed to discuss the curves of radiation re-
sistance of variously proportioned antennae when employed at various
wavelengths relative to the natural wavelength. As preliminary,
the resistance due to radiation from the flat-topped portion of the
antennae is first computed. The equation for this is the summation
of terms in (107) containing the small r's as factors; that is,

$$R_\Omega = \frac{1}{\sin^2(q/2)} \left\{ r_2 A^2 + r_3 A^3 - r_4 A^4 - r_5 A^5 + r_6 A^6 + \cdots \right\} \qquad (109)$$

due to
flat-top

in which

$$A = \frac{2\pi a}{\lambda}$$

$$B = \frac{2\pi b}{\lambda}$$

$$q = \frac{\pi \lambda_0}{\lambda} = 2(A + B) \cdot$$

Since the coefficients (small r's) are functions of B only, as given in
Table II, it follows that when A and B are given, the value of the flat-
top R may be computed. The results of the computations for various
values of A and B are plotted in Figure 11.

In this figure values of B are the abscissae, while the flat-top
resistances in ohms are ordinates. The separate curves numbered
.1, .2, .3, etc. to .9 are for values of $A = .1, .2, .3$, etc. to .9.

The outside end-points of these several curves, through which a
limiting curve is drawn, are determined by the equality of the im-
pressed wavelength λ and the natural wavelength of the antenna λ_0;
that is, by the value of $A + B = \pi/2$, which is the largest value
$A + B$ can have for the fundamental oscillation of the antenna.

22. Curves of Total Radiation Resistance. — The next
step consists in computing the radiation resistance of the vertical
portion of the antenna, using the first three terms of equation (107),
and employing a large number of values of A and B. To these values
of resistance due to the vertical portion of the antenna the correspond-
ing resistance of the flat-top are added so as to give the total resistance

of the antenna for various values of A and B.   Curves of resistance
for various values of $A + B$ are then plotted in Figure 12, with
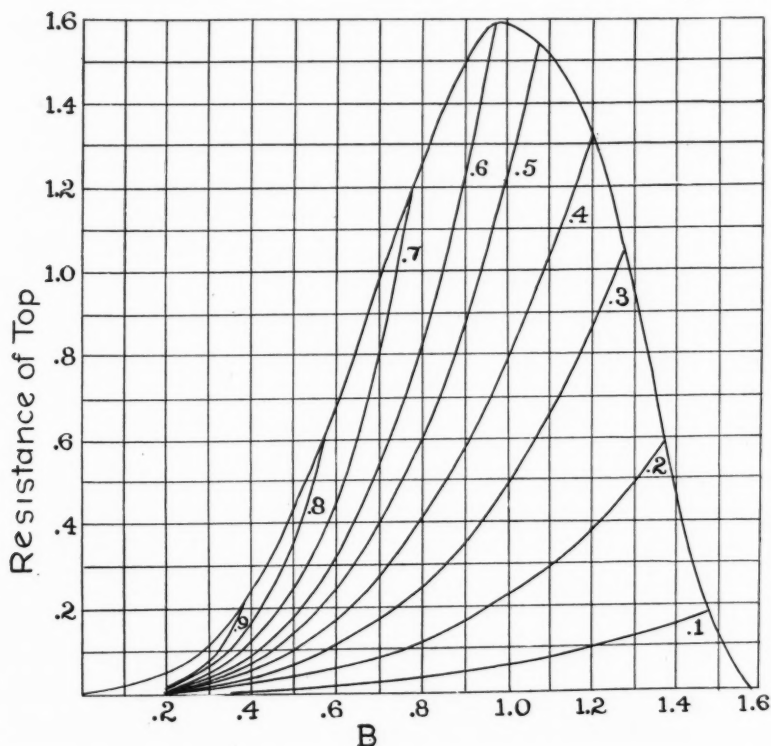values of B as abscissae and values of resistance as ordinates.   Figure

FIGURE 11.   Resistance of horizontal top portion of antenna plotted against
values of B.   The separate curves numbered •1, •2, •3, etc. to •9 are for values
of A = •1, •2, •3, etc. to •9.

13 is an enlarged view of some of the curves that are on too small a
scale to read in Figure 12.   Then to make the family of curves more
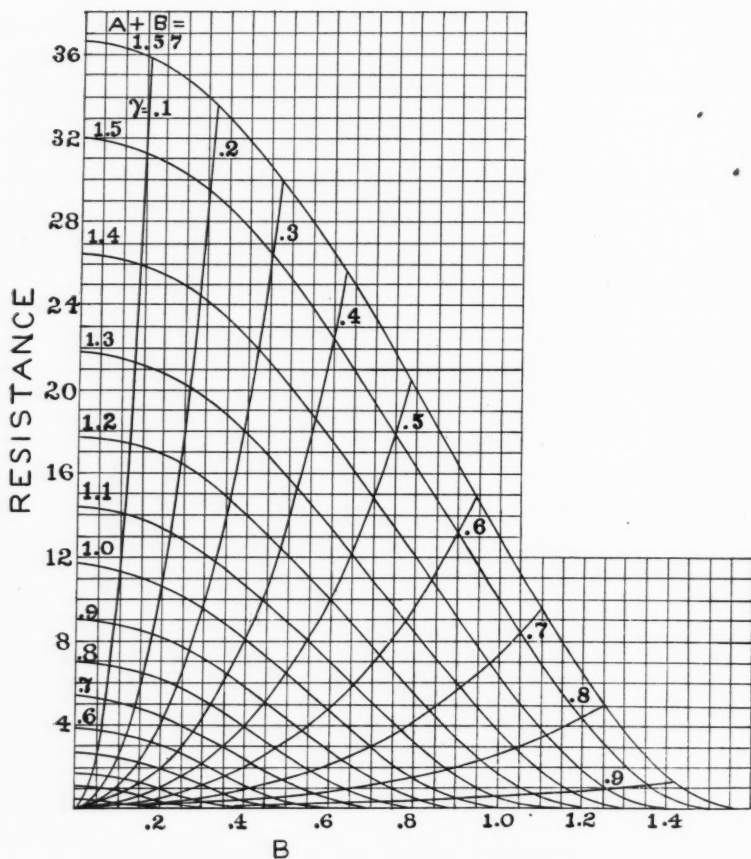useful for ready reference a series of curves are drawn through all

FIGURE 12. Total radiation resistance plotted against values of B. The separate curves through the origin are for designated values of γ. Separate curves not passing through origin are for different values of A + B.

the points which have a common ratio of length of flat-top to length
of total antenna.   This ratio is designated by γ, where

$$\gamma = \frac{B}{A+B} = \frac{b}{a+b} \tag{110}$$

with $b = $ length of flat-top

             $a = $ length of vertical part of antenna.

These γ-curves all pass through the origin.

Next as a final step the curves of Figure 14 are taken from the curves
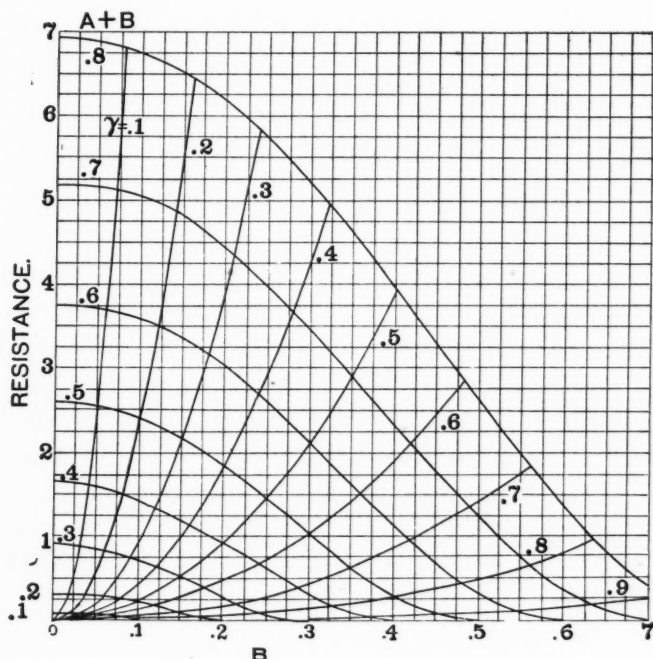

FIGURE 13.   Enlarged view of some of the curves of Figure 12.

of Figures 12 and 13 with the new set of parameters.   These curves of
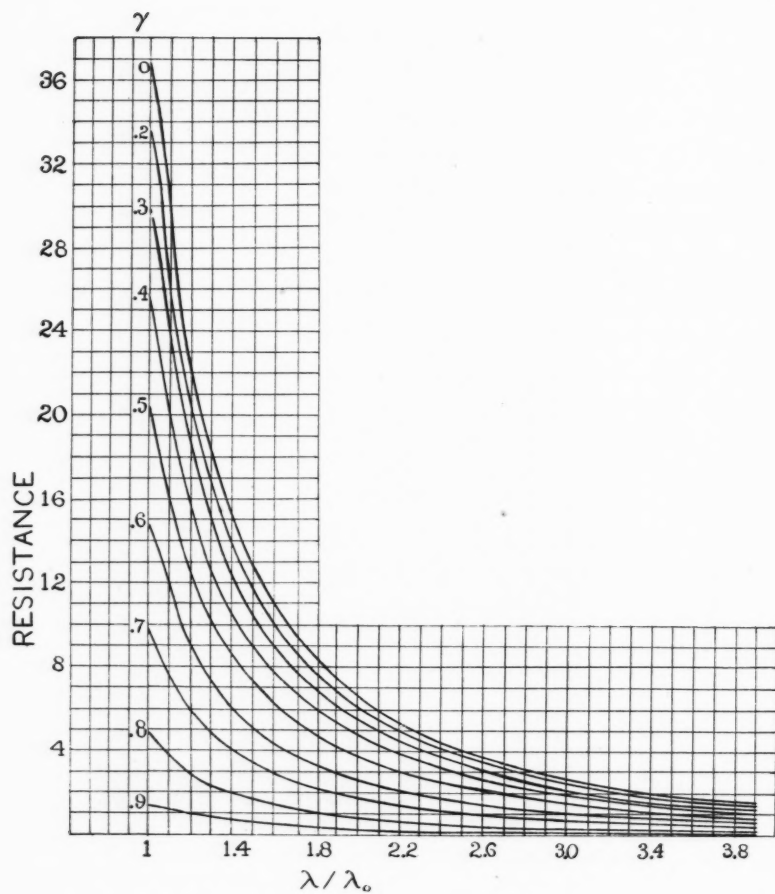Figure 14 are the final curves of total radiation resistance, and are in

FIGURE 14.   Total Radiation resistance plotted against λ/λ_0.   The separate curves marked 0, •2, •3, etc. are for values of $\gamma = 0$, •2, •3, etc.

terms of the ratio of the wavelength employed to the natural wave-length (that is λ/λ_0) and the ratio of the length of flat-top to total length of antenna (that is γ).   Figure 15 is merely a magnified view of certain of the curves that are too small to read on Figure 14.
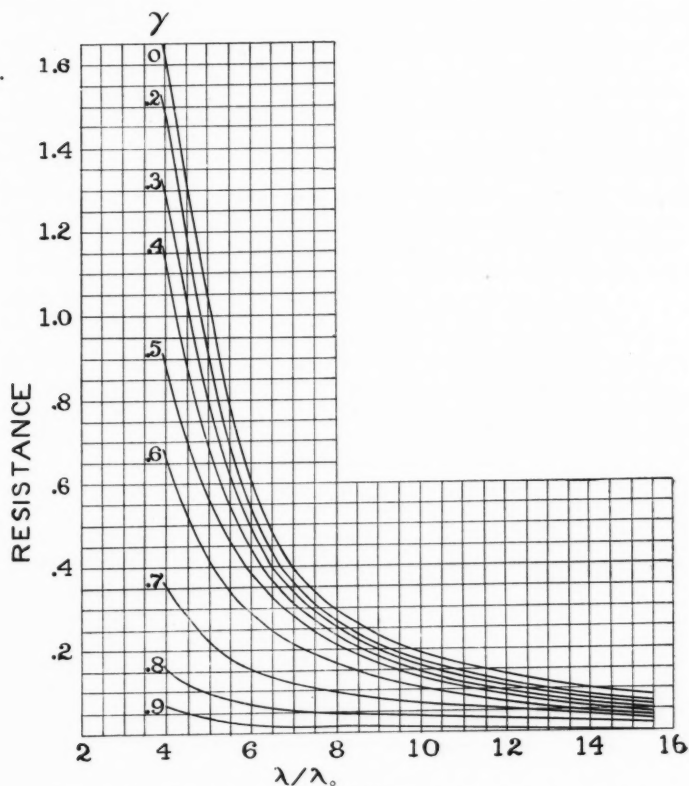

FIGURE 15.   Extension of curves of Figure 14 to larger values of λ/λ_0.

23. Total Radiation Resistance of a Straight Vertical Antenna at Various Wavelengths Obtained by Inductance at the Base.—As an example, let it be required to find the total radiation

TABLE III.

RESISTANCE OF A STRAIGHT VERTICAL ANTENNA FOR DIFFERENT VALUES OF
WAVELENGTH OBTAINED BY INDUCTANCE AT THE BASE.

λ/λ_0 Ratio of Wavelength to Natural Wavelength	R Radiation Resistance in Ohms Computed by Present Theory	Radiation Resistance in Ohms Computed on Doublet Theory
1.00	36.57	98.7
1.12	26.40	78.7
1.21	21.70	67.3
1.31	17.65	57.5
1.43	14.28	48.2
1.57	11.62	40.0
1.74	9.10	32.6
1.97	6.92	25.4
2.24	5.19	19.7
2.62	3.78	14.4
3.14	2.58	10.0
3.93	1.65	6.40
5.26	.90	3.60
7.85	.30	1.16
15.70	.082	.40
31.42	.01	.10

resistance of a straight vertical antenna for various wavelengths
obtained by adding various inductances at the base. For this case
$\gamma = 0$, and from the $\gamma = 0$ curve of Figures 14 and 15 R may be
directly read. The values which were used in plotting this curve are
given in Table III, where they are compared with the corresponding

values computed on the assumption that the oscillator is a Hertzian doublet.   This latter assumption gives

$$R = 160 \frac{\pi^2 a^2}{\lambda^2}$$

It is seen that the departure of the present theory from the doublet theory is very large for the straight vertical antenna, as should be expected.

It should be noted that the first value in the column of resistances computed by the present theory agrees with the value for this case computed by Abraham in the work cited in the introduction.   This one value is the only value arrived at by Abraham for the fundamental oscillation, and is the case of a straight vertical antenna oscillating with its natural frequency.   Abraham's other computed values are for the harmonic vibrations with more than one loop of potential always without loading the antenna by inductance, and without any flat-top extension of the antenna.

24.   Comparison of Computations on the Present Theory with Dr. Austin's Values for the Battleship " Maine."—Figure 16 gives the Radiation Resistance of the Antenna of the Battleship "Maine" as computed by the present Theory in comparison with Dr. Austin's measured values of the total resistance of this antenna, and in comparison with values computed on the doublet theory of Hertz.   The black dots of Figure 16 are Dr. Austin's observed values. The heavy line was obtained by computation by the present theory, and the weaker line, by computation regarding the antenna as a doublet of half-length equal to the vertical height of the antenna.

It is seen that the departure between the present theory and the doublet theory is not so great as in the case of the straight vertical antenna, for the reason that the doublet theory becomes more nearly correct as the half-length of the oscillator becomes small in comparison with the wavelength.

Neither of the theories gives a rising value of the resistance with increase of wavelength, and, as Dr. Austin has pointed out, his rising values for long waves are probably not due to radiation from the antenna but possibly to dielectric hysteresis in the ground beneath the flat-top.

I do not give more extended comparisons with experimental values at the present time, because I am now making some experiments to

see how much reliance may be placed in antenna resistance measurements made by buzzer methods of excitation in comparison with measurements made by excitation with gaseous oscillators and other methods of continuous excitation.
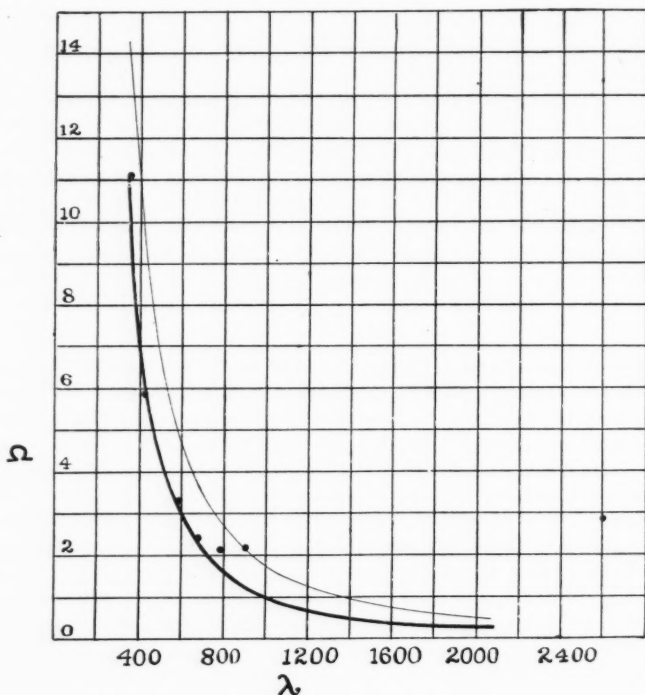

FIGURE 16. Total radiation resistance versus wavelength for the Antenna of the Battleship "Maine." Black dots are Dr. Austin's observed values; heavy line, computations by present theory; light line, computations by doublet theory.

25. Example of Different Methods of Constructing an Antenna that Will Have a Specified Resistance for a Given Wavelength.

— Let it be required to construct an antenna that will have a given resistance (4 ohms, say) for a given wavelength (2000

meters, say). To solve this problem, it is only necessary to look up the four ohm point on the different γ-curves of Figures 14 or 15, and find the corresponding value of λ/λ_0. We can then find the λ_0 of the antenna, since λ is given. Dividing the λ_0 by 4 we obtain the total length of antenna. The value of γ gives the fractional part of this length which is to be horizontal. The complete result is tabulated in Table IV.

TABLE IV.

CONSTANTS OF THE DIFFERENT ANTENNAE THAT HAVE 4 OHMS RESISTANCE AT 2000 METERS.

γ	λ/λ_0	λ_0	Total Length Meters	Vertical Length Meters	Horizontal Length Meters	Intensity Factor in Horizontal Plane
.8	1.075	1860	465	93.0	372	.275
.7	1.39	1435	359	107.7	251.3	.300
.6	1.67	1198	299	119.6	179.4	.310
.5	1.94	1030	258	129.0	129.0	.312
.4	2.18	916	229	137.4	91.6	.313
.3	2.32	861	215	150.5	64.5	.314
.2	2.44	820	205	164.0	41.0	.315
.0	2.52	793	198	198.0	00.0	.320

The question as to which of these antenna to choose for the given purpose is now chiefly a problem in economics. The economic question is, which, for example, is cheaper: Two poles or towers 93 meters high and 372 meters apart, or one tower 198 meters high? This of course pre-supposes that it is designed to use a flat-top antenna instead of some other type, such as an umbrella.

The problem is, however, not wholly economic because the lower antenna would be preferable as a receiving antenna on account of its weaker response to atmospheric disturbances. There is also the further question as to which of the tabulated antennae will give the greatest vertical intensity of electric and magnetic force on the horizon at a distant receiving station. This is the subject matter of the next Part (Part V).

PART V.

FIELD INTENSITIES AND SUMMARY.

26.   The Electric and Magnetic Intensities at a Distant Point in the Horizontal Plane. — Equation (19) gives the values of the electric and magnetic intensities at a distant point due to the vertical portion of the antenna.   If we replace I of that equation by its value in terms of I_0 from equation (6), and make $\cos\ \theta = 0$, we have the intensities in the horizontal plane in terms of I_0, which is the amplitude of the current at the base of the antenna.   This gives

$$E_\theta = H_\phi = \frac{2I_0}{cr_0} \cos \frac{2\pi}{\lambda}\ (ct - r_0) \left[\frac{\cos B - \cos G}{\sin \dfrac{\pi\lambda_0}{2\lambda}} \right]. \qquad (111)$$

The quantities outside the square brackets are constant for a given distance r_0 and a given amplitude of transmitting current I_0.   The *relative intensities* are therefore determined by the factor in the square brackets, which we may designate by

$$X = \frac{\cos B - \cos G}{\sin \dfrac{\pi\lambda_0}{2\lambda}}. \qquad (112)$$

Using the values of B, G, given in equation (20) and the value of γ in (110), this equation (112) becomes

$$X = \frac{\cos \gamma \left(\dfrac{\pi\lambda_0}{2\lambda} \right) - \cos \dfrac{\pi\lambda_0}{2\lambda}}{\sin \dfrac{\pi\lambda_0}{2\lambda}}. \qquad (113)$$

This quantity X we shall call "The Intensity Factor in the Horizontal Plane."   It is to be noted that the electric and magnetic intensities in the horizon plane are not effected by radiation from the flat-top; for, by equation (55), the field intensities from the flat-top are zero for $z = 0$; that is, all over the horizontal plane through the origin.

In Figure 17 the Intensity Factor in the Horizontal Plane is plotted for various values of γ and various values of λ/λ_0. Taking from these curves the values of the intensity factors corresponding to the values
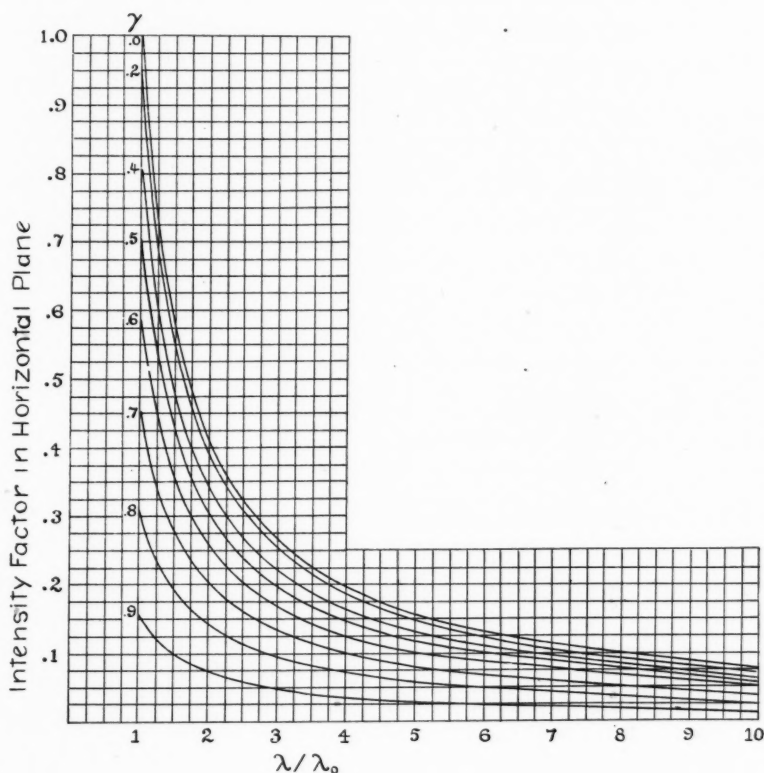

FIGURE 17.   Relative intensity of vertical component of Electric Force in a horizontal plane at a given distance from various antennae and for a given amplitude of transmitting current.

of γ and λ/λ_0 of Table IV we obtain the results in the last column of Table IV.   It is seen that the intensity factor is slightly smaller for

the larger values of the relative length of flat-top. This diminished value of the intensity factor should be compensated by the use of a slightly larger transmitting current. The required increase of current may be easily computed by equation (111).

27. Summary. —This paper contains a mathematical theory of the flat-top antenna. The process employed consists in the integration of the effects of an aggregate of doublets assumed to be distributed along the antenna so as to give a current distribution described by equation (1) and illustrated in Figure 2. The electric and magnetic field intensity due to each of the doublets is determined by the Maxwell and Hertz Theories for all distant points in space. These field intensities are summed up for all the doublets with strict allowance for the differences of phase due to different doublets; the summation gives the resultant field intensities. Then by Poynting's theorem the power radiated from the antenna through a distant hemisphere (bounded by the earth's surface assumed plane) is computed by the integration of a number of intricate expressions. From the radiated power the radiation resistance is obtained by dividing by the mean square of the current at the base of the antenna. Tables of coefficients for computing radiation resistance are given, and curves are plotted of the calculated values of radiation resistance for different ratios of the length of the flat-top to the total length of the antenna and for different relative wavelengths obtained by loading the antenna with inductance. Curves are also given for determining the relative electric and magnetic field intensities in the horizontal plane for differently proportioned antennae variously loaded. Various equations developed in the treatment may find application to problems in the design of radiotelegraphic stations. Although this investigation was undertaken in ignorance of a simple case investigated by Professor Max Abraham, by a similar fundamental method, his work was discovered early in the course of the treatment and served as a check on one of the resistance values here given. This paper may be regarded as an extension of the remarkable work of Professor Abraham.

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